Navier - Stokes Eqns

$$\left.\begin{aligned} \rho \frac{du}{dt} &= \rho B_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\ \rho \frac{dv}{dt} &= \rho B_y - \frac{\partial p}{\partial y} + \mu (\partial^2 v/\partial x^2 + \cdots \\ \rho \frac{dw}{dt} &= \rho B_z - \frac{\partial p}{\partial z} + \mu (\partial^2 w/\partial x^2 + \cdots \end{aligned}\right\}$$

$$\text{Or } \rho \frac{DV_x}{Dt} = \rho B_x + \frac{\partial p}{\partial x} + \mu \nabla^2 V_x$$

[The non-linearity is present in the substantial derivative, wherein one finds terms such as $V_x (\partial V_x/\partial x)$, etc]

$$\left[\frac{D\vec{V}}{Dt} = V_x \frac{\partial \vec{V}}{\partial x} + V_y \frac{\partial \vec{V}}{\partial y} + V_z \frac{\partial \vec{V}}{\partial z} + \frac{\partial V}{\partial t} \right]$$

Notes and

Points from Lecture notes

B. S. Mahara Rao

20/7/39

Also points from Pauli's book on Relativity

26/9/68 ie 29 years later

BSM

26/9/68

[* From the scheme we at once obtain 4 $\S X$, by ~~adding~~ subtracting a_{12}, a_{21}.
Hence $\S$ is obtained. Similarly we can obtain η and ζ in succession]

Whittaker §7: The dirn cosines of $\angle Q$ being $\propto (\S-c)\cos\beta - (\eta-b)\cos\gamma$, etc

the actual dirn cosines are

$$\frac{(\S-c)\cos\beta - (\eta-b)\cos\gamma}{\sqrt{\{(\S-c)\cos\beta - (\eta-b)\cos\gamma\}^2 + \ldots}}$$

& using Lagrange's identity the expⁿ under the square root

$$= \{(\S-a)^2 + (\eta-b)^2 + (\zeta-c)^2\}\{\cos^2\alpha + \cos^2\beta + \cos^2\gamma\} - \{(\S-a)\cos\alpha + (\eta-b)\cos\beta + \ldots$$

$$= AR^2 - AK^2 = KR^2$$

$\therefore$ Dirn cosines are $\dfrac{(\S-c)\cos\beta - (\eta-b)\cos\gamma}{KR}$, etc, ..

This is equivalent to what is given in the text but is more straightforward being analytical.

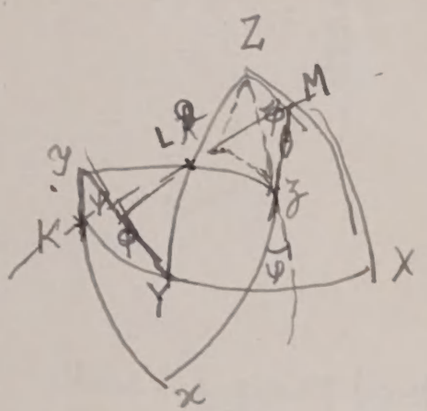
15/7/42

Eulerian angles and Eulerian parameters: (Whittaker § 11).

(1) For the rotation of a rigid body through an angle ω about an axis $(\cos\alpha, \cos\beta, \cos\gamma)$ the Eulerian parameters are defined by

$$\left.\begin{aligned} \xi &= \cos\alpha \sin\tfrac{1}{2}\omega \\ \eta &= \cos\beta \sin\tfrac{1}{2}\omega \\ \zeta &= \cos\gamma \sin\tfrac{1}{2}\omega \\ \chi &= \cos\tfrac{1}{2}\omega \end{aligned}\right\} \qquad (1)$$

The $(\alpha, \beta, \gamma, \omega)$ might be called Chasles's parameters. It is required to express the Eulerian parameters in terms of the Eulerian angles θ, ϕ, ψ



There are 2 alternative methods:

(1) purely analytical, by comparing the schemes of transformation involving the Eulerian parameters, and the angles

(2) Purely geometrical, by deducing that angle $R\hat{Z}z$ (where R is the centre of rotation) is equal to $\frac{1}{2}(\pi - \phi - \psi)$, based on the several positions of axes Rz after rotations ϕ, θ, ψ about R, K and z.

(1) is mentioned in Whittaker, (2) worked out. Both are a little laborious. An alternative method of combining both (1) & (2) is suggested here. Equate the spurs of the two matrices of transformation. This gives at once

$$4\chi^2 - 1 = 4\cos^2\tfrac{1}{2}\theta \cos^2\tfrac{1}{2}(\phi+\psi) - 1$$

$$\text{ie} \quad \chi = \cos\tfrac{1}{2}\theta \cos\tfrac{1}{2}(\phi+\psi)$$

$$\text{ie} \quad \cos\tfrac{1}{2}\omega = \cos\tfrac{1}{2}\theta \cos\tfrac{1}{2}(\phi+\psi). \qquad *$$

The resultant of rotations (ξ, η, ζ, χ) and $(\xi', \eta', \zeta', \chi')$: § 9. Whittaker.

It is remarked in the text that it is "readily seen" that $(\xi'', \dots)$ are given by

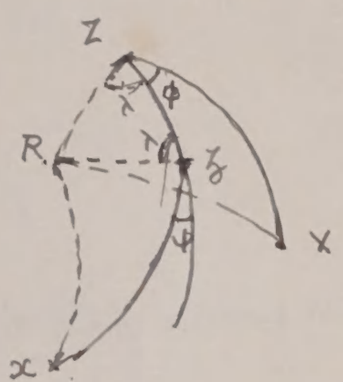
$$\xi'' = \xi\chi' + \eta\zeta' - \zeta\eta' + \chi\xi', \text{ etc.}.$$

If this is meant as the result of the multiplication of the matrices of the two transformations, the procedure is too laborious.

A reference to the triangle isosceles triangle RZz where $Z\hat{R}z = \omega$, $Zz = \theta$ shows at once that $RZz = 90° - \frac{1}{2}(\phi + \psi)$. The rest of the work is purely geometrical using spherical triangles as in (2).

(2) Another purely geometrical method much simpler than Whittaker's —

This is suggested by Euler's Theoria motus corporum solidorum, 978 (Vide Todhunter-Leathem. p 228).



If R be the centre of rotation & ω the angle of rotation, this rotation takes Z to z and X to x ie triangle RZX to Rzx which are therefore congruent. ie Hence $R\hat{z}x = R\hat{Z}X$

ie $\lambda + \phi = 180° - (\lambda + \psi)$

ie $\lambda = 90° - \frac{1}{2}(\phi + \psi)$.

Thus the going back to Euler makes things very much simpler than Whittaker's method which considers three rotations about Z, K and z. In this K does not come in at all, and the method definitely superior.

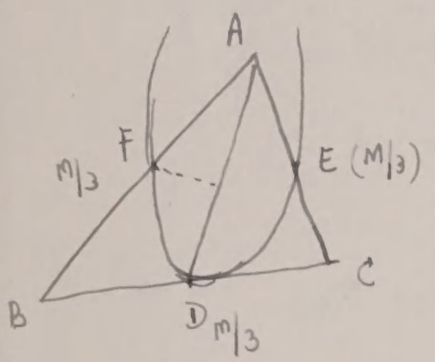
Rotations through the Orthocentre:

momental ellipse at vertex of a triangle: The problem in Loney is:

To show that there is a momental ellipse at A which passes through the middle points F, D of AB, AC and touches BC at its midpoint.

This gives rise to the geometrical theorem: Prove that the conic having A for centre and touching BC at D and passing through F also passes through

the point D. or alternatively: If a conic having A for centre passes through D, E, F it touches BC at D. In the second form the proof is immediate because, A being the centre, AD and EF are conjugate directions and since D is on the conic & BC ∥ to EF BC is a tangent to the conic.



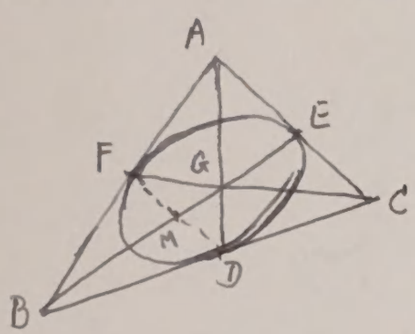
To prove that this is a momental ellipse it is enough to find moments of inertia about any three lines thro' A say AD, AE, AF

$$I_{AD} = \frac{2M}{3} \cdot \frac{\frac{1}{2}\Delta^2}{AD^2} = \frac{M\Delta^2}{6} \cdot \frac{1}{AD^2}$$

$$I_{AF} = \frac{2M}{3} \cdot \frac{\frac{1}{2}\Delta^2}{AB^2} = \frac{M\Delta^2}{6} \cdot \frac{1}{AF^2}$$

$$I_{AE} = \frac{M\Delta^2}{6} \cdot \frac{1}{AE^2}$$ which proves the result.

The next problem is about momental ellipse at G. This leads to the geometrical theorem that the conic touching the sides at E and F, and passing through D touches BC at D and has G for its centre.



Let M be midpoint of DF; since $(DF, MP_\infty) = -1$, polar of L passes thro' P_∞ (on DF). Also polar of E passes thro' P_∞ (on AC ∥ DF) ie P_∞ is pole of LE ie pole of BE lies on l_∞ ie pole of l_∞ lies on BE or BE passes through the centre, similarly CF passes thro' the centre. Thus G is the centre of the conic. That the conic touches BC at D now follows just as in the above theorem. To show that this is a momental ellipse at G, we can proceed as above and show similarly

$$I_{GD} : I_{GE} : I_{GF} = \frac{1}{GD^2} : \frac{1}{GE^2} : \frac{1}{GF^2}.$$

Lamb. p. 128. Ex. 16. This is worked out in Routh. p. 88.

From eqns $A\omega_1^2 + \cdot = T$ (1)

$A^2\omega_1^2 + \cdot = H^2$ (2)

$\omega_1^2 + \cdot = \omega^2$ (3)

$$\omega_1^2 = \frac{BC}{(A-C)(A-B)}(\omega^2 - \lambda_1),$$

$$\lambda_1 = \frac{T(B+C) - H^2}{BC},$$

From (3) $\omega \frac{d\omega}{dt} = \omega_1 \frac{d\omega_1}{dt} + \cdot$

$= \omega_1 \omega_2 \omega_3 \frac{B-C}{A}$, from Euler's eqns.

$= \sqrt{(\lambda_1 - \omega^2)(\lambda_2 - \omega^2)(\lambda_3 - \omega^2)}$.

$AT - G^2 = B(A-B)\omega_2^2 + C(A-C)\omega_3^2$, eliminating ω_1 from (1) and (2)

$A > B > C$, shows $AT - G^2 > 0$, since $CT - G^2 < 0$.

$$\lambda_1 = \frac{T(B+C-A) + (AT-G^2)}{BC} > 0,$$ using $B + C > A$ –

Since λ_2, λ_3 are +ve whose numerators are > that of λ_1

$ABC(\lambda_1 - \lambda_2) = (CT - G^2)(A-B)$. ie $\lambda_1 < \lambda_2$, since $\lambda_3 < \lambda_2$

ie λ_2 is the max.

From this follows easily Whittaker Ex. 28, p. 173.

If $\omega_1 = \text{const}$, $\omega = \text{const}$ also

also if $B = C$, $\omega^2 = \lambda_2 = \lambda_3$ and $\omega_2 = \omega_3$.

Lamb Ex. 20. This amounts to actual verification.

Whittaker §105 : Principal of least curvature : The following is a simpler investigation

From D'Alembert's principle

$$\Sigma \{(m_r \ddot{x}_{r0} - X_r)\delta x_r + \dots\} = 0 \qquad (1)$$

Let constraints be denoted by $\Sigma x_{kr}\, dx_r = 0$ ie $\Sigma x_{kr}\dot{x}_r = 0$

~~if $\delta \Sigma x_{kr}\dot{x}_r = 0$, $\Sigma x_{kr}\delta$~~

$$\Sigma x_{kr}\ddot{x}_r + \Sigma\Sigma \frac{\partial x_{kr}}{\partial x_s}\dot{x}_r\dot{x}_s = 0$$

Taking variation of this, since $\delta x_r = \delta \dot{x}_r = 0$, we have

$$\Sigma x_{kr}\delta\ddot{x}_r = 0 \quad \text{ie } \delta\ddot{x}_r \propto \delta x_r \text{ is a displacement}$$

consistent with geometrical conditions of constraint. Hence from (1)

$$\Sigma (m_r \ddot{x}_{r0} - X_r)\delta\ddot{x}_r = 0$$

This is the condition that

$$\delta\left[\Sigma m_r \left\{\ddot{x}_r - \frac{X_r}{m_r}\right\}^2 + \dots\right] = 0 \text{ for the actual path.}$$

———— v ————

Prange : §6. Solns of Lagrange's eqn

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\rho}\right) - \frac{\partial T}{\partial q_\rho} = Q_\rho$$

Gives $q_\rho = q_\rho(t, \alpha_1 \dots \alpha_n, \beta_1 \dots \beta_n)$ ie ∞^{2n} space-time curves in M_{n+1} or ∞^{2n} trajectories in M_n. Corresponding to every space-time curve there is a trajectory & conversely.

Through a point in M_{n+1} there pass a n-parametric bundle, because of the $2n$-constants, n can be found from initial conds $q_r = q_{r0}$ for $t = t_0$.

In a given dirn ie corresponding to ratios $dq_1 : dq_2 : \dots dq_n$: if there is only one curve. for M_n on the other hand, direction is specified by $dq_1 : \dots dq_n$ and

Weitzenböck: Ency. d. Math III. D. 10. C (19). Kovariant differentiation

$$a_{i_1 i_2 \ldots i_m (i)} = \frac{\partial a_{i_1 i_2 \ldots i_m}}{\partial x_i} - \Gamma^{\lambda}_{i i_1} a_{\lambda i_2 \ldots i_m} - \Gamma^{\lambda}_{i i_2} a_{i_1 \lambda \ldots i_m}$$

$$- \Gamma^{\lambda}_{i i_m} a_{i_1 i_2 \ldots i_{m-1} \lambda}$$

This form is easier to remember.

How to Consider

$\ddot{q}_\rho + \Gamma^{\rho}_{\mu\nu} \dot{q}_\mu \dot{q}_\nu$ as covariant derivative of $\dot{q}_\rho$?

$\dot{q}^\rho$ is a contravariant component & its covariant derivative is given by

$$\frac{\partial \dot{q}^\rho}{\partial q^\alpha} + \Gamma^{\rho}_{\lambda\alpha} \dot{q}^\alpha .$$

Multiplying by $\dot{q}^\lambda$ we have $\ddot{q}^\rho + \Gamma^{\rho}_{\lambda\alpha} \dot{q}^\alpha \dot{q}^\lambda$ as the derivative w.r.t. t

ie acc^n is $\left[\dot{q}^\rho_{(2)} \cdot \dot{q}^\lambda \right]$

———— 0 ————.

and these ratios do not determine the remaining n arbitrary constants but only $n-1$ of them ie there ∞^1 trajectories in a given dirn, but ∞^n through a point (taking $n-1$ ∞^1 possible directions) just as in M_{n+1} also there are ∞^n through a point (one each in $^n\infty^{n-1}$ possible directions).

In case $Q_\rho = \partial_\rho (V_1 - V_n)$ and $T = a_{\kappa\lambda} \dot{q}_\kappa \dot{q}_\lambda$, space-time curves are transformed into each other by a parallel-transformation w.r.t the t-axis (as follows from the nature of Lagrange's eqns itself) and these parallel curves lie on a cylinder M_2 with generators parallel to t-axis. To all these corresponds only one trajectory viz the intersection of M_2 with M_n ie ∞^{2n-1} trajectories alone are possible. Each trajectory might be considered travelled along in ∞^1 ways and from one way of description might be derived all other ways because the time at which a certain point of the trajectory is reached might be multiplied by a constant number.

In such a case let

$$T = \frac{1}{2} g_{\rho\sigma} \dot{q}_\rho \dot{q}_\sigma \quad , \quad g_{\rho\sigma} = g_{\rho\sigma}(q_1 \cdots q_n)$$

Through the kinetic energy we can correspond a line element

$$ds^2 = 2T dt^2 = g_{\rho\sigma} dq_\rho dq_\sigma$$

in a M_n which is now an n-dimensional Riemann-space. We differentiate between covariant & contravariant components of the same vector. For the velocity vector $\dot{q}_\rho$ (more correctly $\dot{q}^\rho$) are the contravariant components. The covariant components are given by $g_{\rho\sigma} \dot{q}^\sigma = \frac{\partial T}{\partial \dot{q}^\rho} = p_\rho$ ie components of generalized momenta. In this sense $Q_1 \cdots Q_n$ are covariant components of impressed forces & the contravariant components are $g^{\rho\sigma} Q_\sigma = Q^\rho$. Forming with the q^ρ's the differential equations

$$T = \frac{1}{2} g_{\lambda\mu} \dot{q}_\lambda \dot{q}_\mu$$

$$* \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\lambda}\right) - \frac{\partial T}{\partial q_\lambda}$$

$$= \frac{d}{dt}\left(g_{\lambda\mu}\dot{q}_\mu\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial q_\lambda}\dot{q}_\mu \dot{q}_\nu$$

$$= g_{\lambda\mu}\ddot{q}_\mu + \frac{\partial g_{\lambda\mu}}{\partial q_\nu}\dot{q}_\nu \dot{q}_\mu - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial q_\lambda}\dot{q}_\mu \dot{q}_\nu$$

$$= g_{\lambda\mu}\ddot{q}_\mu + \left(\frac{1}{2}\frac{\partial g_{\mu\lambda}}{\partial q_\nu} + \frac{1}{2}\frac{\partial g_{\nu\lambda}}{\partial q_\mu}\right)\dot{q}_\mu \dot{q}_\nu - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial q_\lambda}\dot{q}_\mu \dot{q}_\nu \quad \left(\text{interchange of summing indices } \mu \text{ & } \nu\right)$$

$$Q_\lambda = g_{\lambda\mu}\ddot{q}_\mu + \Gamma_{\mu\nu,\lambda}\dot{q}_\mu \dot{q}_\nu \qquad \text{(covariant components of acceleration).}$$

$$g^{\rho\lambda} Q_\lambda = g^{\rho\lambda} g_{\lambda\mu}\ddot{q}_\mu + g^{\rho\lambda}\Gamma_{\mu\nu,\lambda}\dot{q}_\mu \dot{q}_\nu$$

$$Q^\rho = \ddot{q}_\rho + \Gamma^\rho_{\mu\nu}\dot{q}_\mu \dot{q}_\nu \quad \text{· contravariant expression of Lagrange's eq}^{n}_{s}$$

———— o ————

$\frac{dq_1}{Q^1} = \cdots = \frac{dq_n}{Q^n}$, we determine in R_n of $(q_1 \cdots q_n)$ a $(n-1)$-parametric group of curves viz the lines of force of the impressed forces with its tangent in the direction of the impressed force at the point of R_n concerned.

The left hand side of Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\lambda}\right) - \frac{\partial T}{\partial q_\lambda} = g_{\lambda\rho}\ddot{q}_\rho + \left[\begin{matrix}\sigma\tau \\ \lambda\end{matrix}\right]\dot{q}_\sigma \dot{q}_\tau \quad *$$

are said to represent the covariant components of the acceleration vector. Going from covariant to contravariant components on both sides of Lagrange's equation in the form

$$g_{\lambda\rho}\ddot{q}_\rho + \left[\begin{matrix}\sigma\tau \\ \lambda\end{matrix}\right]\dot{q}_\sigma \dot{q}_\tau = Q_\lambda$$

we get $g^{\mu\lambda} g_{\lambda\rho}\ddot{q}_\rho + g^{\mu\lambda}\left[\begin{matrix}\sigma\tau \\ \lambda\end{matrix}\right]\dot{q}_\sigma \dot{q}_\tau = g^{\mu\lambda} Q_\lambda$

i.e. get $g^{\mu\lambda} g_{\rho\lambda}\ddot{q}_\rho + g^{\mu\lambda}\Gamma_{\rho\sigma,\lambda}\dot{q}_\sigma \dot{q}_\tau = g^{\mu\lambda} Q_\lambda$

i.e. $g_{\lambda\mu}\ddot{q}_\mu + \Gamma_{\mu\nu,\lambda}\dot{q}_\mu \dot{q}_\nu = Q_\lambda$

$$g^{\rho\lambda} g_{\lambda\mu}\ddot{q}_\mu + g^{\rho\lambda}\Gamma_{\mu\nu,\lambda}\dot{q}_\mu \dot{q}_\nu = g^{\rho\lambda} Q_\lambda$$

$$a^\rho = \ddot{q}_\rho + \Gamma^\rho_{\mu\nu}\dot{q}_\mu \dot{q}_\nu$$ Contravariant form of Lagrange's eq^n

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For the Calculation

(13/12/68) Lanczos proves (i) & (ii) by merely stating that they follow from the Bianchi identity ~~which, he shows, can be written~~ viz

$$\frac{\partial R_{\alpha\beta\mu\nu}}{\partial x_\rho} + \frac{\partial R_{\alpha\beta\nu\rho}}{\partial x_\mu} + \frac{\partial R_{\alpha\beta\rho\mu}}{\partial x_\nu} = 0$$

which may be written for the special case of 4 dimensions in the form

$$\frac{\partial R_{ik\alpha\beta}}{\partial x_\gamma} \delta^{\alpha\beta\gamma m} = 0.$$

Paper by Lanczos in Ann of Math, 39 (1938), p. 842. A remarkable property of the Riemann-Christoffel tensor.

$$I_1 = R_{\alpha\beta} R^{\alpha\beta}$$

$$I_2 = R^2 \quad (R = g^{ik} R_{ik})$$

$$\delta^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{g}} \epsilon^{\alpha\beta\mu\nu}$$ (ϵ being antisymmetric tensor of 4th order).

introduces.

$$^{*}R^{mn}_{ik} = R_{ik\alpha\beta} \delta^{\alpha\beta mn}$$

$$^{**}R^{ikmn} = R_{\alpha\beta\mu\nu} \delta^{\alpha\beta ik} \delta^{\mu\nu mn}$$

$\delta^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{g}} \epsilon^{\alpha\beta\mu\nu}$

where $\epsilon^{\alpha\beta\mu\nu}$ = +1 for even permutation of the four indices
= −1 for odd ...
= 0, if two indices are equal.

$$k_1 = R_{\alpha\beta\mu\nu} {}^{*}R^{\alpha\beta\mu\nu}$$

$$k_2 = R_{\alpha\beta\mu\nu} {}^{**}R^{\alpha\beta\mu\nu}.$$

Puts Bianchi's identity in the form

$$\frac{\partial R_{ik\alpha\beta}}{\partial x_\gamma} \delta^{\alpha\beta\gamma m} = 0$$

[$\frac{\partial}{\partial x_\gamma}$ denotes covariant differentiation]

Proves Statz the relations

(i) $\frac{\partial {}^{*}R^{\alpha}_{ikm}}{\partial x_\alpha} = 0$, and (ii) $\frac{\partial {}^{**}R^{\alpha}_{ikm}}{\partial x_\alpha} = 0$.

we prove these relations here. Q

(i) To prove that ${}^{*}R^{\alpha\beta}_{ikm,\beta} = 0$

ie $\left(R_{ikp\sigma}\,\delta^{p\sigma\alpha\beta}\right)_{,\beta} = 0.$

or $\dfrac{\partial R_{ikp\sigma}}{\partial x_\beta}\,\delta^{p\sigma\alpha\beta} + R_{ikp\sigma}\left(\delta^{p\sigma\alpha\beta}\right)_{,\beta} = 0$

The first term is zero because of Bianchi's identity. Consider

$$\delta^{p\sigma\alpha\beta}{}_{,\beta} = \left(\frac{1}{\sqrt{g}}\,\epsilon^{p\sigma\alpha\beta}\right)_{,\beta}$$

$$= \epsilon^{p\sigma\alpha\beta}\frac{\partial}{\partial x_\beta}\left(\frac{1}{\sqrt{g}}\right) + \frac{1}{\sqrt{g}}\,\epsilon^{p\sigma\alpha\beta}{}_{,\beta} \qquad \text{- - - (1)}$$

$$\epsilon^{p\sigma\alpha\beta}{}_{,\beta} = \Gamma^{p}_{\lambda\beta}\,\epsilon^{\lambda\sigma\alpha\beta} + \Gamma^{\sigma}_{\lambda\beta}\,\epsilon^{p\lambda\alpha\beta} + \Gamma^{\alpha}_{\lambda\beta}\,\epsilon^{p\sigma\lambda\beta} + \Gamma^{\beta}_{\lambda\beta}\,\epsilon^{p\sigma\alpha\lambda} \quad \text{- - -},$$

The first three terms vanish because of the symmetry in λ, β of the Christoffel symbols and the antisymmetry of tensor $\epsilon^{\cdots}$; and the fourth term can be written out by interchange of λ and β giving

$$\epsilon^{p\sigma\alpha\beta}{}_{,\beta} = \Gamma^{\lambda}_{\beta\lambda}\,\epsilon^{p\sigma\alpha\beta} \qquad (2)$$

~~The~~ From the relation $\dfrac{1}{\sqrt{g}}\dfrac{\partial(\sqrt{g})}{\partial x_\beta} = \Gamma^{\lambda}_{\beta\lambda}$ we can write the first term of (1) in the form $-\epsilon^{p\sigma\alpha\beta}\,\dfrac{1}{g}\,\dfrac{\partial(\sqrt{g})}{\partial x_\beta}$ and hence, using (2), (1) reduces to

$$-\epsilon^{p\sigma\alpha\beta}\,\Gamma^{\lambda}_{\beta\lambda} + \Gamma^{\lambda}_{\beta\lambda}\,\epsilon^{p\sigma\alpha\beta}$$

which is equal to zero. Hence the result is proved.

$$\delta^{\alpha\beta\mu\nu} = -\delta^{\mu\beta\alpha\nu} = \delta^{\mu\nu\alpha\beta}$$

$$\delta_{ik} \to \delta_{ii} = 0 \to 0 = 0 ?$$

R

To prove (ii) ie $\dfrac{\partial {}^{**}R^{\alpha}_{ikm}}{\partial x_\alpha} = 0$.

$${}^{**}R^{\alpha}_{ikm} = g_{i\beta}\, g_{k\rho}\, g_{m\sigma}\, {}^{**}R^{\alpha\beta\rho\sigma}$$

$$= g_{i\beta}\, g_{k\rho}\, g_{m\sigma}\, R_{\mu\nu\gamma\delta}\, \delta^{\alpha\beta\mu\nu}\, \delta^{\gamma\delta\rho\sigma}$$

ie to prove that $$\left(R_{\mu\nu\gamma\delta}\, \delta^{\alpha\beta\mu\nu}\, \delta^{\gamma\delta\rho\sigma}\right)_{,\alpha} = 0 \qquad (1)$$

Left hand side of (1) =

$$\frac{\partial}{\partial x_\alpha}\left(R_{\mu\nu\gamma\delta}\, \delta^{\alpha\beta\mu\nu}\right)\delta^{\gamma\delta\rho\sigma} + \frac{\partial}{\partial x_\alpha}\left(R_{\mu\nu\gamma\delta}\right)\delta^{\alpha\beta\mu\nu}\,\delta^{\gamma\delta\rho\sigma} + R_{\mu\nu\gamma\delta}\, \delta^{\alpha\beta\mu\nu}\cdot \delta^{\gamma\delta\rho\sigma}{}_{,\alpha}$$

1st term $\dfrac{\partial}{\partial x_\alpha}\left(R_{\gamma\delta\mu\nu}\, \delta^{\mu\nu\alpha\beta}\right)\delta^{\gamma\delta\rho\sigma}$

$$= \frac{\partial}{\partial x_\alpha}\left(R_{i\kappa\mu\nu}\, \delta^{\mu\nu\alpha\beta}\right)\delta^{i\kappa\rho\sigma}$$

$= 0$, from what is proved in (i) above.

Second term $= R_{\mu\nu\gamma\delta}\, \delta^{\alpha\beta\mu\nu}\cdot \delta^{\gamma\delta\rho\sigma}{}_{,\alpha}$ — — — (2)

$$= R_{\mu\nu\alpha\delta}\, \delta^{\gamma\beta\mu\nu}$$

$$\delta^{\gamma\delta\rho\sigma}{}_{,\alpha} = \left(\frac{1}{\sqrt{g}}\epsilon^{\gamma\delta\rho\sigma}\right)_{,\alpha} = \frac{\partial}{\partial x_\alpha}\left(\frac{1}{\sqrt{g}}\right)\epsilon^{\gamma\delta\rho\sigma} + \frac{1}{\sqrt{g}}\,\epsilon^{\gamma\delta\rho\sigma}{}_{,\alpha}$$

$$R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{hj}}{\partial x_i^i \partial x_k^k} + \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} - \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ij}}{\partial x^h \partial x^j} \right)$$

$$+ g^{\alpha\beta} \left(\Gamma_{\alpha, hj} \Gamma_{\beta, ik} - \Gamma_{\alpha, hk} \Gamma_{\beta, ij} \right)$$

$i \neq k, \; m \neq n$

$(i,k) = (1,2)(13)(14)(23)(24)(34)$ } 36 values - ie R^{**}_{ikmn} has 36 components.

$(m,n) =$ "

$$1^{st}\text{ term} = -\frac{1}{g}\frac{\partial(\sqrt{g})}{\partial x_\alpha}\epsilon^{\nu\delta\rho\sigma}$$

$$= -\frac{1}{\sqrt{g}}\Gamma^{\lambda}_{\alpha\lambda}\epsilon^{\nu\delta\rho\sigma}$$

$$2^{nd}\text{ term} = \frac{1}{\sqrt{g}}\left\{\Gamma^{\nu}_{\delta\alpha}\epsilon^{\lambda\delta\rho\sigma} + \Gamma^{\delta}_{\lambda\alpha}\epsilon^{\nu\lambda\rho\sigma} + \Gamma^{\rho}_{\lambda\alpha}\epsilon^{\nu\delta\lambda\sigma} + \Gamma^{\sigma}_{\lambda\alpha}\epsilon^{\nu\delta\rho\lambda}\right\}$$

Multiplying by $\delta^{\alpha\beta\mu\nu}$ ie $\frac{1}{\sqrt{g}}\epsilon^{\alpha\beta\mu\nu}$ and interchanging α and λ, (2) becomes omitting $R_{\mu\nu\rho\sigma}$ and $\frac{1}{g}$

$$\left(-\Gamma^{\alpha}_{\delta\alpha}\epsilon^{\nu\delta\rho\sigma} + \Gamma^{\nu}_{\alpha\lambda}\epsilon^{\alpha\delta\rho\sigma} + \Gamma^{\delta}_{\alpha\lambda}\epsilon^{\nu\alpha\rho\sigma} + \Gamma^{\rho}_{\alpha\lambda}\epsilon^{\nu\delta\alpha\sigma} + \Gamma^{\sigma}_{\alpha\lambda}\epsilon^{\nu\delta\rho\alpha}\right)\epsilon^{\lambda\beta\mu\nu}R_{\mu\nu\rho\delta}$$

$$= 0$$

$$K_2 = R_{\alpha\beta\mu\nu}{}^{**}R^{\alpha\beta\mu\nu} \qquad \left[{}^{**}R^{ikmn} = R_{\alpha\beta\mu\nu}\delta^{\alpha\beta ik}\delta^{\mu\nu mn}\right.$$

Relation between $K_2; I_3; I_{11}; I_2$:

Give to $\alpha\beta\mu\nu$ the values in Eddington's scheme.

1212	1223	1313	1324	1423	2323	2424
1213	1224	1314	1334	1424	2324	2434
1214	1234	1323	1414	1434	2334	3434

with $1234 - 1324 + 1423 = 0$.

$$\begin{aligned}
{}^{**}R^{ikmn} = {} & R_{1212}\delta^{12ik}\delta^{12mn} + R_{1213}\delta^{12ik}\delta^{13mn} + R_{1214}\delta^{12ik}\delta^{14mn} \\
& + R_{1223}\delta^{12ik}\delta^{23mn} + R_{1224}\delta^{12ik}\delta^{24mn} + R_{1234}\delta^{12ik}\delta^{34mn} \\
& + R_{1313}\delta^{13ik}\delta^{13mn} + R_{1314}\delta^{13ik}\delta^{14mn} + R_{1323}\delta^{13ik}\delta^{23mn} \\
& + R_{1324}\delta^{13ik}\delta^{24mn} + R_{1334}\delta^{13ik}\delta^{34mn} + R_{1414}\delta^{14ik}\delta^{14mn} \\
& + R_{1423}\delta^{14ik}\delta^{23mn} + R_{1424}\delta^{14ik}\delta^{24mn} + R_{1434}\delta^{14ik}\delta^{34mn} \\
& + R_{2323}\delta^{23ik}\delta^{23mn} + R_{2324}\delta^{23ik}\delta^{24mn} + R_{2334}\delta^{23ik}\delta^{34mn}
\end{aligned}$$

$$+ R_{2424}\,\delta^{24\,dc\;24\,mn} + R_{2434}\,\delta^{24\,dc\;34\,mn} + R_{3434}\,\delta^{34\,dc\;34\,mn}$$

$$I_2 = R^2 = g^{\alpha\beta}\, g^{\alpha'\beta'}\, g^{\gamma\delta}\, g^{\gamma'\delta'}\, R_{\alpha\beta\gamma\delta}\, R_{\alpha'\beta'\gamma'\delta'}$$

$$I_1 = g^{\alpha\beta}\, g^{\alpha'\beta'}\, g^{\gamma\delta'}\, g^{\delta\delta'}\, R_{\alpha\beta\gamma\delta}\, R_{\alpha'\beta'\delta'\delta'}$$

$$I_3 = g^{\alpha\alpha'}\, g^{\beta\beta'}\, g^{\gamma\gamma'}\, g^{\delta\delta'}\, R_{\alpha\beta\gamma\delta}\, R_{\alpha'\beta'\gamma'\delta'}$$

$$K_1 = g^{\alpha\alpha'}\, g^{\beta\beta'}\, \delta^{\gamma\delta\gamma'\delta'}\, R_{\alpha\beta\gamma\delta}\, R_{\alpha'\beta'\gamma'\delta'}$$

$$K_2 = \delta^{\alpha\beta\alpha'\beta'}\, \delta^{\gamma\delta\gamma'\delta'}\, R_{\alpha\beta\gamma\delta}\, R_{\alpha'\beta'\gamma'\delta'}$$

$$K_2 - I_3 + 4 I_1 - I_2 = 0.$$

$$\delta^{\alpha\beta\alpha'\beta'}\,\delta^{\gamma\delta\gamma'\delta'} - g^{\alpha\alpha'} g^{\beta\beta'} g^{\gamma\gamma'} g^{\delta\delta'} + 4\, g^{\alpha\beta} g^{\alpha'\beta'} g^{\gamma\gamma'} g^{\delta\delta'} - g^{\alpha\beta} g^{\alpha'\beta'} g^{\gamma\delta} g^{\gamma'\delta'} = 0.$$

$$\delta^{\alpha\beta\alpha'\beta'}\,\delta^{\gamma\delta\gamma'\delta'} = \delta^{\alpha\beta}_{\lambda\mu}\,\delta^{\alpha'\beta'}_{\lambda'\mu'}\,\delta^{\gamma\delta}_{\nu\rho}\,\delta^{\gamma'\delta'}_{\nu'\rho'}$$

$\alpha\beta\ \gamma\delta\ \alpha'\beta'\gamma'\delta'$

(12,12)(12,12)

(12,12)(12,13)

(12,13)(12,13)

(12,12)(12,34)

(12,13)(12,34)

(12,34)(12,34)

I. Characterisations of canonical systems.

20/6/43.

(1) If for the system

$$\frac{dq_r}{dt} = A_r \; ; \; \frac{dp_r}{dt} = B_r \qquad (A_r, B_r = f^{ns} \text{ of } q_1 \ldots q_n, p_1 \ldots p_n, t) \tag{1}$$

If the multiplier $m = 1$ is the system canonical ?

$\frac{\partial A_r}{\partial q_1} + \Sigma\left(\frac{\partial B_r}{\partial q_r} + \frac{\partial q_r}{\partial p_r}\right) = 0$ when $m = 1$, $\Sigma\left(\frac{\partial q_r}{\partial q_r} + \frac{\partial p_r}{\partial p_r}\right) = 0$.

This is equivalent to the question:

$$\text{If } \underbrace{\iint \cdots \int}_{2n} \delta p_1 \ldots \delta p_n \delta q_1 \ldots \delta q_n = \text{const} \tag{2}$$

be an integral invariant of the above system, is the system canonical ?

We knows that from

$$\Omega^{(1)} = \Sigma\left(\delta p_\rho^{(\mu)} \delta q_\rho^{(\mu)} - \delta q_\rho^{(\lambda)} \delta p_\rho^{(\mu)}\right)$$

a differential form for the Hamiltonian system, we can build by Cartan's process $\Omega^{(2)}, \Omega^{(3)}, \ldots \Omega^{(n)}$ all of which give integral invariants.

It is proved in Whittaker § 116 that if $\iint \Omega'' = \iint \Sigma \delta p_\rho \delta q_\rho$ be an integral invariant for system (1) then (1) is ~~Hamiltonian~~ canonical in form. Now

$$\Omega^{(n)} = \begin{vmatrix} \delta p_1^{(1)} \; \delta q_1^{(1)} & \cdots & \delta p_n^{(1)} \delta q_n^{(1)} \\ \cdots & \cdots & \cdots \\ \delta p_1^{(2n)} & \cdots & \delta q_n^{(2n)} \end{vmatrix} . \tag{3}$$

by proper choice of $\delta p^{(1)}, \delta q^{(1)}$ [See Poarge p. 670. footnote 264] this can be brought to the form in integrand of (2). If (2) be an integral invariant we can show that $\underbrace{\iint \cdots \int}_{2n} \Omega^{(n)}$ must also be another so because $\delta p_1 \ldots \delta p_n \delta q_1 \ldots \delta q_n$ as well as $\Omega^{(n)}$ express the volume element

in the $2n$-dimensional space R_{2n} in two different ways. The question therefore is:

Is (1) canonical if (3) gives an integral invariant?

That this is so if $\iint \Omega^{(1)} = C$ is an integral invariant is true. Let us try

with $\Omega^{(2)}$ ie it is given that

$$\iiiint \sum_{\rho,\lambda} \begin{vmatrix} \delta p_\rho^{(1)} & \delta q_\rho^{(1)} & \delta p_\lambda^{(1)} & \delta q_\lambda^{(1)} \\ \delta p_\rho^{(4)} & \delta q_\rho^{(4)} & \delta p_\lambda^{(4)} & \delta q_\lambda^{(4)} \end{vmatrix} \text{ is an integral invariant}$$

or as in Whittaker Ex. 2, p. 286 that

$$\iiiint \Sigma\, \delta q_i\, \delta p_i\, \delta q_j\, \delta p_j \text{ is an integral invariant}$$

or (using the summation convention) that

$$\frac{d}{dt} \frac{\partial(q_i, q_j, p_i, p_j)}{\partial(\lambda_1, \lambda_2, \mu_1, \mu_2)} = 0$$

$$\frac{\partial(Q_i, Q_j, p_i, p_j)}{\partial(\quad)} + \frac{\partial(q_i, Q_j, p_i, p_j)}{\partial(\quad)} + \frac{\partial(q_i, q_j, P_i, p_j)}{\partial(\quad)} + \frac{\partial(q_i, q_j, p_i, P_j)}{\partial(\quad)} = 0.$$

(omitting the denominators)

ie $$\frac{\partial Q_i}{\partial q_K} \frac{\partial(q_K, q_j, p_i, p_j)}{\partial(\quad)} + \frac{\partial Q_i}{\partial p_K} \frac{\partial(p_K, q_j, p_i, p_j)}{\partial(\quad)}$$

$$+ \frac{\partial Q_j}{\partial q_K} \frac{\partial(q_i, q_K, p_i, p_j)}{\partial(\quad)} + \frac{\partial Q_j}{\partial p_K} \frac{\partial(q_i, p_K, p_i, p_j)}{\partial(\quad)}$$

$$+ \frac{\partial P_i}{\partial q_K} \frac{\partial(q_i, q_j, q_K, p_j)}{\partial(\quad)} + \frac{\partial P_i}{\partial p_K} \frac{\partial(q_i, q_j, p_K, p_j)}{\partial(\quad)}$$

$$+ \frac{\partial P_j}{\partial q_K} \frac{\partial(q_i, q_j, p_i, q_K)}{\partial(\quad)} + \frac{\partial P_j}{\partial p_K} \frac{\partial(q_i, q_j, p_i, p_K)}{\partial(\quad)} = 0.$$

By suitable manipulation of dummy indices this can be arranged as

$$\left(\frac{\partial Q_i}{\partial q_K} + \frac{\partial P_K}{\partial p_i}\right) \frac{\partial(q_K, q_j, p_i, p_j)}{\partial(\quad)} + \left(\frac{\partial Q_j}{\partial q_K} + \frac{\partial P_K}{\partial p_j}\right) \frac{\partial(q_i, q_K, p_i, p_j)}{\partial(\quad)}$$

$$+\left(\frac{\partial P_i}{\partial q_K}-\frac{\partial P_K}{\partial q_i}\right)\frac{\partial(q_i,q_j,q_K,p_j)}{\partial(\quad)}+\left(\frac{\partial Q_i}{\partial p_K}-\frac{\partial Q_K}{\partial p_i}\right)\frac{\partial(p_K,q_j,p_i,p_j)}{\partial(\quad)}=0.$$

and the first two terms can be combined to give $2\left(\frac{\partial Q_i}{\partial q_K}+\frac{\partial P_K}{\partial p_i}\right)\frac{\partial(q_K,q_j,p_i,p_j)}{\partial(\quad)}$

Since coeffk of $\frac{\partial q_K}{\partial\lambda_1}\frac{\partial p_i}{\partial\lambda_3}\frac{\partial q_j}{\partial\lambda_2}\frac{\partial p_j}{\partial\lambda_3}$ for every individual j must vanish we have

Similarly

$$\left.\begin{aligned}\frac{\partial Q_i}{\partial q_K}+\frac{\partial P_K}{\partial p_i}&=0\\ \frac{\partial P_i}{\partial q_K}-\frac{\partial P_K}{\partial q_i}&=0\\ \frac{\partial Q_i}{\partial p_K}-\frac{\partial Q_K}{\partial p_i}&=0\end{aligned}\right\}$$

These prove that the eqns are of the canonical form.

Let us next treat $\Omega^{(3)}$ ie

$$\frac{d}{dt}\frac{\partial(q_i,q_j,q_k,p_i,p_j,p_K)}{\partial(\lambda_1,\lambda_2,\lambda_3,\mu_1,\mu_2,\mu_3)}=0.$$

This gives twelve terms:

$$\frac{\partial Q_i}{\partial q_k}\partial(q_k,q_j,q_K,p_i,p_j,p_K)+\frac{\partial Q_i}{\partial p_l}\partial(p_l,q_j,q_K,p_i,p_j,p_K)$$
$$+\frac{\partial Q_j}{\partial q_l}\partial(q_i,q_l,q_K,p_i,p_j,p_K)+\frac{\partial Q_j}{\partial p_l}\partial(q_i,p_l,q_K,p_i,p_j,p_K)$$
$$+\frac{\partial Q_K}{\partial q_l}\partial(q_i,q_j,q_l,p_i,p_j,p_K)+\frac{\partial Q_K}{\partial p_l}\partial(q_i,q_j,p_l,p_i,p_j,p_K)$$
$$+\frac{\partial P_i}{\partial q_l}\partial(q_i,q_j,q_K,q_l,p_j,p_K)+\frac{\partial P_i}{\partial p_l}\partial(q_i,q_j,q_K,p_l,p_j,p_K)$$
$$+\frac{\partial P_j}{\partial q_l}\partial(q_i,q_j,q_K,p_i,q_l,p_K)+\frac{\partial P_j}{\partial p_l}\partial(q_i,q_j,q_K,p_i,p_l,p_K)$$
$$+\frac{\partial P_K}{\partial q_l}\partial(q_i,q_j,q_K,p_i,p_j,q_l)+\frac{\partial P_K}{\partial p_l}\partial(q_i,q_j,q_K,p_i,p_j,p_l)=0$$

From combining the 1st term with the 2nd, 3rd with the 10th, 5th with 12th etc. get by manipulating dummy indices

$$3\left(\frac{\partial Q_i}{\partial q_l}+\frac{\partial P_l}{\partial p_i}\right)\partial(q_l, q_j, q_k, p_i, p_j, p_k)$$

Second term

from the Since the coeffn of $\frac{\partial p_l}{\partial \lambda_1}\frac{\partial p_i}{\partial \mu_1}\frac{\partial q_j}{\partial \lambda_2}\cdot\frac{\partial q_k}{\partial \lambda_3}\frac{\partial p_j}{\partial \mu_2}\frac{\partial p_k}{\partial \mu_3}$ for cyclic particular value of j & k should

be zero we get $\frac{\partial Q_i}{\partial p_l}-\frac{\partial Q_l}{\partial p_i}=0$ by interchanging i & l

The same result is given from the 4th & 6th terms. Similarly the 7th, 9th & 11th terms gives $\frac{\partial P_i}{\partial q_l}-\frac{\partial P_l}{\partial q_i}=0.$

Thus the case $\Omega^{(3)}$ is treated like $\Omega^{(1)}$ in Whittaker. our treatment for $\Omega^{(2)}$ is slightly different but it can be carried in just the same way as for $\Omega^{(1)}$ & $\Omega^{(3)}$. The method can be extendedde to any $\Omega^{(r)}$. This gives

$$\frac{d}{dt}\frac{\partial(q_{i_1}, q_{i_2}, \ldots q_{i_r}, p_{i_1}, p_{i_2}, \ldots p_{i_r})}{\partial(\lambda_1, \lambda_2, \ldots \lambda_r, \mu_1, \mu_2 \ldots \mu_r)}=0.$$

Typical terms in the expansion are

$$\frac{\partial Q_{i_j}}{\partial q_{i_K}}\frac{\partial(q_{i_1}, \ldots q_{i_{j-1}}, q_{i_K}, q_{i_{j+1}}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_r})}{\partial(\lambda_1 \ldots \lambda_r, \mu_1 \ldots \mu_r)}$$

$$\frac{\partial Q_{i_j}}{\partial p_{i_K}}\frac{\partial(q_{i_1}, \ldots q_{i_{j-1}}, p_{i_K}, q_{i_{j+1}}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_r})}{\partial(\lambda_1 \ldots \lambda_r, \mu_1 \ldots \mu_r)}$$

$$\frac{\partial P_{i_j}}{\partial q_{i_K}}\frac{\partial(q_{i_1}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_{j-1}}, q_{i_K}, p_{i_{j+1}}, \ldots p_r)}{\partial(\lambda_1 \ldots \lambda_r, \mu_1 \ldots \mu_r)}$$

and $\dfrac{\partial P_{i_j}}{\partial p_{i_K}} \dfrac{\partial(q_{i_1}, \ldots q_{i_r}, p_{i_1} \ldots p_{i_{j-1}}, p_{i_K}, p_{i_{j+1}}, \ldots p_{i_r})}{\partial(\lambda_1 \ldots \lambda_r, \mu_1 \ldots \mu_r)}$

The first and fourth can be taken together & written

$$\left(\frac{\partial Q_{i_j}}{\partial q_{i_K}} + \frac{\partial P_{i_K}}{\partial p_{i_j}}\right) \frac{\partial(q_{i_1}, \ldots q_{i_{j-1}}, q_{i_K}, q_{i_{j+1}}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_r})}{\partial(\lambda_1 \ldots \lambda_r, \mu_1 \ldots \mu_r)}$$

Similarly from the second term, equating to zero the coefficient of interchanging i_j & i_K the Jacobian of the terms being the coefficient of the

$\dfrac{\partial p_{i_K}}{\partial \lambda_K} \dfrac{\partial p_{i_j}}{\partial \mu_j}$ ..... Jacobian factor of the term becomes

$$\frac{\partial(q_{i_1}, \ldots q_{i_{j-1}}, p_{i_j}, q_{i_{j+1}}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_{j-1}}, p_{i_K}, p_{i_{j+1}}, \ldots p_{i_r})}{\partial(\lambda_1 \ldots \lambda_r, \mu_1 \ldots \mu_r)}.$$

and interchange of p_{i_j} with p_{i_K} in this determinant means an interchange of columns equal in number to $(j-1)+(r-j)$

$(j-1) + (r-j+1) + (r-j) + (j-1) = 2r-1$ is an odd number

and hence equating the coefficient of

$\dfrac{\partial p_{i_K}}{\partial \lambda_K} \dfrac{\partial p_{i_j}}{\partial \mu_j}$ ..... to zero gives

$$\frac{\partial Q_{i_j}}{\partial p_{i_K}} - \frac{\partial Q_{i_K}}{\partial p_{i_j}} = 0$$

Similarly 3rd term gives $\dfrac{\partial P_{i_j}}{\partial q_{i_K}} - \dfrac{\partial P_{i_K}}{\partial q_{i_j}} = 0.$

Thus the system is canonical when $\int\int \Omega^{(r)}$ is an integral invariant.
$\widetilde{2r}$

In the case of $\Omega^{(n)}$ we have

$$\frac{d}{dt}\,\frac{\partial(q_1 \cdots q_n, p_1 \cdots p_n)}{\partial(\lambda_1 \cdots \lambda_n, \mu_1 \cdots \mu_n)} = 0, \quad \text{without any summation.}$$

Typical terms are

$$\frac{\partial Q_i}{\partial q_k}\,\frac{\partial(q_1 \cdots q_{i-1}, q_k, q_{i+1} \cdots q_n, p_1 \cdots p_n)}{\partial(\lambda_1 \cdots \lambda_n, \mu_1 \cdots \mu_n)}$$

$$\frac{\partial Q_i}{\partial p_k}\,\frac{\partial(q_1 \cdots q_{i-1}, p_k, q_{i+1} \cdots q_n, p_1 \cdots p_n)}{\partial(\lambda_1 \cdots \lambda_n, \mu_1 \cdots \mu_n)}$$

$$\frac{\partial P_i}{\partial q_k}\,\frac{\partial(q_1 \cdots q_n, p_1 \cdots p_{i-1}, q_k, p_{i+1} \cdots p_n)}{\partial(\lambda_1 \cdots \lambda_n, \mu_1 \cdots \mu_n)}$$

$$\frac{\partial P_i}{\partial p_k}\,\frac{\partial(q_1 \cdots q_n, p_1 \cdots p_{i-1}, p_k, p_{i+1} \cdots p_n)}{\partial(\lambda_1 \cdots \lambda_n, \mu_1 \cdots \mu_n)}.$$

The first term which is a summation over k, gives the single term $\frac{\partial Q_i}{\partial q_i}\,\frac{\partial(q_1 \cdots q_n, p_1 \cdots p_n)}{\partial(\lambda_1 \cdots \mu_n)}$

The second term " also a " $= 0$ for all values of k

Third " " $= 0$ "

The fourth term gives $\frac{\partial P_i}{\partial p_i}\,\frac{\partial(q_1 \cdots p_n)}{\partial(\lambda \cdots \mu_n)}$. Thus $\left(\frac{\partial Q_i}{\partial q_i} + \frac{\partial P_i}{\partial p_i}\right) = 0$ is a summation over the index i

The case of $\Omega^{(n)}$ appears therefore to be a singular case.

Let us now try $\Omega^{(n-1)}$ which can be written

$$\underbrace{\int\cdots\int}_{2(n-1)}(\delta q_2 \cdots \delta q_n\, \delta p_2 \cdots \delta p_n + \delta q_1\, \delta q_3 \cdots \delta q_n\, \delta p_1\, \delta p_3 \cdots \delta p_n + \cdots)$$

which has n terms just like $\sum_{i=1}^{n} \delta p_i\, \delta q_i$.

$$\frac{d}{dt}\left\{ \frac{\partial(q_2 \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)} + \frac{\partial(q_1 q_3 \cdots q_n, p_1, p_3 \cdots p_n)}{\partial(\quad)} + \cdots \right\} = 0. \qquad (A)$$

1st term gives: $\frac{\partial Q_i}{\partial q_K} \frac{\partial(q_2 \cdots q_{i-1}, q_K, q_{i+1} \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)} + \frac{\partial Q_i}{\partial p_K} \frac{\partial(q_2 \cdots q_{i-1}, p_K, q_{i+1}, \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)}$

$i \neq 1$.

$$+ \frac{\partial P_i}{\partial q_K} \frac{\partial(q_2 \cdots q_n, p_2 \cdots p_{i-1}, q_K, p_{i+1} \cdots p_n)}{\partial(\quad)} + \frac{\partial P_i}{\partial p_K} \frac{\partial(q_2 \cdots q_n, p_1 \cdots p_{i-1}, p_K, p_{i+1} \cdots p_n)}{\partial(\quad)}$$

there being a summation over K.

either $i \neq k$.

Here the first term gives non-zero value only when $K = 1$ or $K = i$

i.e. $\sum_{i \neq 1} \frac{\partial Q_i}{\partial q_1} \frac{\partial(q_2 \cdots q_{i-1}, q_1, q_{i+1}, p_2 \cdots p_n)}{\partial(\quad)} + \sum_{i \neq 1} \frac{\partial Q_i}{\partial q_i} \frac{\partial(q_2 \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)}$

Sub. 2nd gives $\sum_{i \neq 1} \frac{\partial Q_i}{\partial p_1} \frac{\partial(q_2 \cdots q_{i-1}, p_1, q_{i+1} \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)} + \sum_{i \neq 1} \frac{\partial Q_i}{\partial p_i} \frac{\partial(q_2 \cdots q_{i-1}, p_i, q_{i+1} \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)}$

3rd gives $\sum_{i \neq 1} \frac{\partial P_i}{\partial q_1} \frac{\partial(q_2 \cdots q_n, p_2 \cdots p_{i-1}, q_1, p_{i+1} \cdots p_n)}{\partial(\quad)} + \sum_{i \neq 1} \frac{\partial P_i}{\partial q_i} \frac{\partial(q_2 \cdots q_n, p_2 \cdots p_{i-1}, q_i, p_{i+1} \cdots p_n)}{\partial(\quad)}$

4th gives $\sum_{i \neq 1} \frac{\partial P_i}{\partial p_1} \frac{\partial(q_2 \cdots q_n, p_2 \cdots p_{i-1}, p_i, p_{i+1} \cdots p_n)}{\partial(\quad)} + \sum_{i \neq 1} \frac{\partial P_i}{\partial p_i} \frac{\partial(q_2 \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)}$

$= \sum_{i \neq 1} \left(\frac{\partial Q_i}{\partial q_i} + \frac{\partial P_i}{\partial p_i} \right) \frac{\partial(q_2 \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)}$ ✱ taking the 1st and last suffixes + etc.

In the first summation take term with $i = 2$ i.e. $\frac{\partial Q_2}{\partial q_1} \frac{\partial(q_1, q_3 \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)}$

Next

Corresponding to this take the term corresponding to the 7th in the second term of (A) with $i = 1$

i.e. $\frac{\partial P_1}{\partial p_2} \frac{\partial(q_1 q_3 \cdots q_n, p_2 \cdots p_n)}{\partial(\quad)}$

Thus we have $\frac{\partial Q_2}{\partial q_1} + \frac{\partial P_1}{\partial p_2} = 0.$

& in general $\frac{\partial Q_i}{\partial q_K} + \frac{\partial P_K}{\partial p_i} = 0 \qquad (i \neq K)$

Also from $\sum \left(\frac{\partial Q_i}{\partial q_i} + \frac{\partial P_i}{\partial p_i} \right) = 0$, i.e. $\sum \left(\frac{\partial Q_i}{\partial q_i} + \frac{\partial P_i}{\partial p_i} \right) = 0$, etc we get

$$\frac{\partial Q_i}{\partial q_i} + \frac{\partial P_i}{\partial p_i} = 0 \qquad \text{for all } i$$

In ~~an identical~~ similar way we can prove the validity of the other conditions

$$\frac{\partial Q_i}{\partial p_k} - \frac{\partial Q_k}{\partial p_i} = 0 \qquad \text{for } i \neq k \text{ & } i = k$$

$$\frac{\partial P_i}{\partial q_k} - \frac{\partial P_k}{\partial q_i} = 0 \qquad \text{for } i \neq k \text{ & } i = k$$

Thus the theorem is true for $\Omega^{(n-1)}$ also.

Examine $\Omega^{(n)}$ in greater detail. For the given system the variational equations are (Prange. p. 694)

$$\left.\begin{aligned} \frac{d\xi_\rho}{dt} &= \frac{\partial Q_\rho}{\partial p_\lambda}\pi_\lambda + \frac{\partial Q_\rho}{\partial q_\lambda}\xi_\lambda \\ \frac{d\pi_\rho}{dt} &= -\frac{\partial P_\rho}{\partial p_\lambda}\pi_\lambda - \frac{\partial P_\rho}{\partial q_\lambda}\xi_\lambda \end{aligned}\right\} \qquad \begin{aligned} \xi_\rho &= \delta q_\rho \\ \pi_\rho &= \delta p_\rho \end{aligned}$$

If $\begin{vmatrix} \delta^{(1)}q_1 & \delta p_1^{(1)} & \cdots & \delta q_n^{(1)} & \delta p_n^{(1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \delta q_1^{(2n)} & \delta p_1^{(2n)} & \cdots & \delta q_n^{(2n)} & \delta p_n^{(2n)} \end{vmatrix} = \text{const}$ be a solution of this system

ie $\begin{vmatrix} \xi_1^{(1)} & \pi_1^{(1)} & \cdots & \xi_n^{(1)} & \pi_n^{(1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi_1^{(2n)} & \pi_1^{(2n)} & \cdots & \xi_n^{(2n)} & \pi_n^{(2n)} \end{vmatrix} = \text{Const. is an integral}$

This perhaps leads to $\sum\left(\frac{\partial Q_r}{\partial x_i} + \frac{\partial P_r}{\partial p_i}\right) = 0$ & we have to search for some other method.

Let us try method in Goursat (Leçons sur le Problème de Pfaff. p. 231) :—

There it is shown that if (§57)

$$I_n = \int F(x_1 \ldots x_n)\, dx_1 \ldots dx_n$$

be an integral invariant F is a last multiplier & further by putting

$$\Omega_{n-1} = X_1 dx_2 \ldots dx_n \pm X_2 dx_3 \ldots dx_n dx_1 \pm \ldots \pm X_n dx_1 \ldots dx_{n-1}$$

that if M be a multiplier $\int M \Omega_{n-1}$ is an integral invariant. Again if

$$\omega_{n-1} = A_1 dx_2 \ldots dx_n \pm A_2 dx_3 \ldots dx_n dx_1 \pm \ldots$$

(all + if n be odd ; $\pm$ if n be even

the necessary & sufficient condition that $\int \omega_{n-1}$ should be an integral invariant is

$$X(A(f)) - A(X(f)) + HA(f) = 0$$

where $A(\ ) = \sum_i A_i \frac{\partial}{\partial x_i}$; $X(\ ) = \sum X_i \frac{\partial}{\partial x_i}$; $H = \sum \frac{\partial X_i}{\partial x_i}$.

That:

$$\text{ie } X_1 \frac{\partial}{\partial x_1}\left(A_1 \frac{\partial f}{\partial x_1} + \ldots\right) + \ldots$$
$$- A_1 \frac{\partial}{\partial x_1}\left(X_1 \frac{\partial f}{\partial x_1} + \ldots\right) - \ldots + H\left(A_1 \frac{\partial f}{\partial x_1} + \ldots\right) = 0 \quad \text{whatever } f$$

If $A_1 = A_2 = \ldots = 1$, then

$$X_r \frac{\partial}{\partial x_r}\left(A_s \frac{\partial f}{\partial x_s}\right) - A_r \frac{\partial}{\partial x_r}\left(X_s \frac{\partial f}{\partial x_s}\right)$$
$$= X_r A_s \frac{\partial^2 f}{\partial x_r \partial x_s} - A_r X_s \frac{\partial^2 f}{\partial x_r \partial x_s} = 0$$

Hence $H = 0$. Thus $H = 0$ is also the necessary & sufficient condition for $dx_2 \ldots dx_n \pm dx_3 \ldots dx_n dx_1 \pm \ldots$

But in the case considered here we have not ω_{n-1} but a ω_{2n-2} related to a ω_{2n}. We have to search another method:

16/7/43. Is it not obvious from the very form that the integral invariancy of $\Omega^{(n)}$ leads immediately to that of Ω, and hence does it not follow that the system is canonical? Obviously this is true: ~~but is not~~

See Goursat. p.230 where it is proved that if $\int \omega_p$ & $\int \omega_q$ are two integral invariants of the system $dx_i/dt = X_i$, the integral $\int \omega_p \omega_q$ is also invariant, where ω_p & ω_q are differential forms of orders p & q and the multiplication is symbolic. The proof is as follows: If the given system be changed under the Poincaré transformation to

$$\frac{dy_1}{0} = \cdots = \frac{dy_{n-1}}{0} = \frac{dz}{1} = dt$$

ω_p & ω_q transform to two symbolic forms Ω_p, Ω_q whose coefficients do not involve z since ω_p & ω_q are integrands of invariant integrals. The same is true of Ω_p Ω_q & hence $\omega_p \omega_q$ of which $\Omega_p \Omega_q$ is the transform is the integrand of an integral invariant. This method gives that $\int (\omega_p)^2$, $\int (\omega_p)^3$, etc are all integral invariants. The converse theorem however is not true in general, but in the special case where $\int \Omega^{(n)}$ or in Goursat's notation $\int (\omega_2)^n$ is an integral invariant, it follows from the above reasoning that $\int \omega_2$ is an invariant ie the system is canonical. The same argument can be applied to any $\int (\omega_2)^r$.

The previous direct method attempted, analogous to Whittaker's method for ω_2, holds for every $\Omega^{(n)}$ or $(\omega_2)^r$ except $r = n$ & this is a very interesting fact in itself.

Korkine's theorem: viz if Poisson's theorem be true for any two integrals $f_1 = c_1$ & $f_2 = c_2$ the system (1) is canonical.

Reference to Korkine is in F.d.m. 1871 not in the Library. We give below two proofs of the theorem:

(1) The proof in Goursat · p. 232 of Poisson's theorem can be employed to deduce the converse also: The following is Goursat's proof.

Let $I_2 = \int \omega_2 = \int \sum_i dx_i\, dp_i$

Then $(\omega_2)^n \equiv n!\, dx_1\, dp_1 \cdots dx_n\, dp_n$ · If f_1 & f_2 be two integrals of the system, the symbolic product $(\omega_2)^{n-1} df_1\, df_2$ is equal to $(f_1, f_2)\, dx_1\, dp_1 \cdots dx_n\, dp_n$ (ibid p. 164) except for a constant factor. Now since if f_1 & f_2 be two integrals of the system df_1 & df_2 are solns of the variation system, & $\int \omega_2$ being an integral invariant, so also are $(\omega_2)^n$ & $(\omega_2)^{n-1}$. Thus $(\omega_2)^n$ & $(\omega_2)^{n-1} df_1\, df_2$ are give integrals of the variational system & so also is

$$\frac{(\omega_2)^{n-1} df_1\, df_2}{(\omega_2)^n}$$

ie (f_1, f_2) is an integral of the variational system & since it does not involve the differentials it is a solution of the original system. This is Poisson's theorem.

To prove the converse we note that, when we are given that (f_1, f_2) is an integral whenever f_1 & f_2 are integrals, it follows that

$\frac{(\omega_2)^{n-1} df_1\, df_2}{(\omega_2)^n}$ is a solution of the variational equations.

Since f_1 & f_2 are integrals so also are df_1 & df_2. Hence it follows that ω_2 also gives a solution of the variational equations ie $\int \omega_2$ is an integral invariant ie the system is canonical.

(2) Direct proof: This proof is found in my M. Sc. Lecture notes for 42-43. Book. 4.

Transformation group property of integrals: If $f = c$ be an integral of system (1) and if

$$\delta q_r = \frac{\partial f}{\partial p_r}\delta\alpha \; ; \; \delta p_r = -\frac{\partial f}{\partial q_r}\delta\alpha$$ be a soln of the variational equations is the system canonical?

The variational equations are given by (Prange. p. 694)

$$\left.\begin{aligned} \frac{d\xi_\rho}{dt} &= \frac{\partial Q_\rho}{\partial p_\lambda}\pi_\lambda + \frac{\partial Q_\rho}{\partial q_\lambda}\xi_\lambda \\ \frac{d\pi_\rho}{dt} &= +\frac{\partial P_\rho}{\partial p_\lambda}\pi_\lambda + \frac{\partial P_\rho}{\partial q_\lambda}\xi_\lambda \end{aligned}\right\}$$

of the above be solutions

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial f}{\partial p_\rho}\right) &= -\frac{\partial Q_\rho}{\partial p_\lambda}\frac{\partial f}{\partial q_\lambda} + \frac{\partial Q_\rho}{\partial q_\lambda}\frac{\partial f}{\partial p_\lambda} \\ \frac{d}{dt}\left(-\frac{\partial f}{\partial q_\rho}\right) &= -\frac{\partial P_\rho}{\partial p_\lambda}\frac{\partial f}{\partial q_\lambda} + \frac{\partial P_\rho}{\partial q_\lambda}\frac{\partial f}{\partial p_\lambda} \end{aligned}\right\} \qquad (1)$$

Also since $f = \text{const}$, is an integral

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_\lambda}\dot{Q}_\lambda + \frac{\partial f}{\partial p_\lambda}P_\lambda = 0. \qquad (2)$$

(1) gives
$$\left.\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial p_\rho}\right) + \frac{\partial}{\partial q_\lambda}\left(\frac{\partial f}{\partial p_\rho}\right)Q_\lambda + \frac{\partial}{\partial p_\lambda}\left(\frac{\partial f}{\partial p_\rho}\right)P_\lambda &= -\frac{\partial Q_\rho}{\partial p_\lambda}\frac{\partial f}{\partial q_\lambda} + \frac{\partial Q_\rho}{\partial q_\lambda}\frac{\partial f}{\partial p_\lambda} \\ \frac{\partial}{\partial t}\left(-\frac{\partial f}{\partial q_\rho}\right) + \frac{\partial}{\partial q_\lambda}\left(-\frac{\partial f}{\partial q_\rho}\right)Q_\lambda + \frac{\partial}{\partial p_\lambda}\left(-\frac{\partial f}{\partial q_\rho}\right)P_\lambda &= -\frac{\partial P_\rho}{\partial p_\lambda}\frac{\partial f}{\partial q_\lambda} + \frac{\partial P_\rho}{\partial q_\lambda}\frac{\partial f}{\partial p_\lambda} \end{aligned}\right\} \qquad (3)$$

(2) gives
$$\left.\begin{aligned} &\frac{\partial}{\partial p_\rho}\left(\frac{\partial f}{\partial t}\right) + \frac{\partial}{\partial p_\rho}\left(\frac{\partial f}{\partial q_\lambda}\right)Q_\lambda + \frac{\partial f}{\partial q_\lambda}\frac{\partial Q_\lambda}{\partial p_\rho} + \frac{\partial}{\partial p_\rho}\left(\frac{\partial f}{\partial p_\lambda}\right)P_\lambda + \frac{\partial f}{\partial p_\lambda}\frac{\partial P_\lambda}{\partial p_\rho} = 0 \\ &-\frac{\partial}{\partial q_\rho}\left(\frac{\partial f}{\partial t}\right) - \frac{\partial}{\partial q_\rho}\left(\frac{\partial f}{\partial q_\lambda}\right)Q_\lambda - \frac{\partial f}{\partial q_\lambda}\frac{\partial Q_\lambda}{\partial q_\rho} - \frac{\partial}{\partial q_\rho}\left(\frac{\partial f}{\partial p_\lambda}\right)P_\lambda - \frac{\partial f}{\partial p_\lambda}\frac{\partial P_\lambda}{\partial q_\rho} = 0 \end{aligned}\right\} \qquad (4)$$

(3) & (4) give by subtraction

$$\left.\begin{aligned} \frac{\partial f}{\partial q_\lambda}\frac{\partial Q_\lambda}{\partial p_\rho}+\frac{\partial f}{\partial p_\lambda}\frac{\partial P_\lambda}{\partial p_\rho} &= \frac{\partial Q_\rho}{\partial q_\lambda}\frac{\partial f}{\partial z_\lambda}-\frac{\partial Q_\rho}{\partial z_\lambda}\frac{\partial f}{\partial q_\lambda} \\ -\frac{\partial f}{\partial q_\lambda}\frac{\partial Q_\lambda}{\partial z_\rho}-\frac{\partial f}{\partial p_\lambda}\frac{\partial P_\lambda}{\partial z_\rho} &= +\frac{\partial P_\rho}{\partial q_\lambda}\frac{\partial f}{\partial z_\lambda}-\frac{\partial P_\rho}{\partial z_\lambda}\frac{\partial f}{\partial q_\lambda} \end{aligned}\right\} \qquad (5)$$

$$\text{ie}\quad \left.\begin{aligned} \frac{\partial f}{\partial z_\lambda}\left(\frac{\partial Q_\lambda}{\partial q_\rho}-\frac{\partial Q_\rho}{\partial q_\lambda}\right)+\frac{\partial f}{\partial q_\lambda}\left(\frac{\partial Q_\rho}{\partial z_\lambda}+\frac{\partial P_\lambda}{\partial q_\rho}\right) &= 0 \\ -\frac{\partial f}{\partial z_\lambda}\left(\frac{\partial P_\rho}{\partial q_\lambda}+\frac{\partial Q_\lambda}{\partial z_\rho}\right)+\frac{\partial f}{\partial q_\lambda}\left(\frac{\partial P_\lambda}{\partial z_\rho}-\frac{\partial P_\rho}{\partial z_\lambda}\right) &= 0. \end{aligned}\right\} \qquad (6)$$

Since this is to be true for any arbitrary integral f, the coefficients of $\frac{\partial f}{\partial z_\lambda}$ & $\frac{\partial f}{\partial q_\lambda} = 0$

Hence the conds making (1) a canonical system.

otherwise: If $f(p, q, t) = \text{const}$, be an integral

$$\frac{\partial f}{\partial p_\lambda}\pi_\lambda+\frac{\partial f}{\partial z_\lambda}x_\lambda = \text{const},$$ is an integral of the variational eqns with the fwcn π_λ, x_λ

$$\text{ie}\quad \frac{\partial f}{\partial p_\lambda}\frac{d\pi_\lambda}{dt}+\frac{d}{dt}\left(\frac{\partial f}{\partial p_\lambda}\right)\pi_\lambda+\cdots = 0$$

$$\text{ie}\quad \frac{\partial f}{\partial p_\lambda}\left(\frac{\partial R_\lambda}{\partial p_\rho}\pi_\rho+\frac{\partial R_\lambda}{\partial z_\rho}x_\rho\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial p_\lambda}\right)\pi_\lambda+\cdots = 0$$

$$+\frac{\partial f}{\partial z_\lambda}\left(\frac{\partial Q_\lambda}{\partial p_\rho}\pi_\rho+\frac{\partial Q_\lambda}{\partial z_\rho}x_\rho\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial z_\lambda}\right)x_\lambda = 0 .$$

$$\text{ie}\quad \pi_\rho\left\{\frac{d}{dt}\left(\frac{\partial f}{\partial p_\rho}\right)+\frac{\partial f}{\partial z_\lambda}\frac{\partial Q_\lambda}{\partial p_\rho}+\frac{\partial f}{\partial q_\lambda}\frac{\partial P_\lambda}{\partial p_\rho}\right\}+x_\rho\left\{\frac{d}{dt}\left(\frac{\partial f}{\partial z_\rho}\right)+\frac{\partial f}{\partial z_\lambda}\frac{\partial Q_\lambda}{\partial z_\rho}+\frac{\partial f}{\partial p_\lambda}\frac{\partial P_\lambda}{\partial z_\rho}\right\} =$$

$$\text{ie}\quad -\frac{\partial f}{\partial q_\rho}\left\{\quad\text{"}\quad\right\}+\frac{\partial f}{\partial p_\rho}\left\{\quad\right\} = 0$$

is identically satisfied in $\frac{\partial f}{\partial z_\lambda}, \frac{\partial f}{\partial q_\lambda}$. This method leads to the same as above & is in no way simpler.

canonical transformations

(1) If the motion along a trajectory is denoted by an infinitesimal canonical transformation the system is canonical.

This is evident from the eqns of the inf. can. trn viz $\Delta q_r = \frac{\partial K}{\partial p_r}\Delta t$; $\Delta p_r = -\frac{\partial K}{\partial q_r}\Delta t$.

(2) If the given system is transformed under a canonical transformation to a canonical system, the original system is canonical, ~~because the inverse of a canonical transformation is canonical~~. Here we define a canonical transformation

$$P_\rho = P_\rho(P_1 \cdots P_n, Q_1 \cdots Q_n);\quad Q_\rho = \Psi_\rho(P_1 \cdots P_n, Q_1 \cdots Q_n)$$

as canonical if $(Q_\sigma, Q_\tau) = 0$, $(\Psi_\sigma, \Psi_\tau) = 0$, $(Q_\sigma, \Psi_\tau) = \delta_{\sigma\tau}$.

From this follows the relations between Lagrange brackets

$$[Q_\sigma, Q_\tau] = 0,\quad [\Psi_\sigma, \Psi_\tau] = 0,\quad [Q_\sigma, \Psi_\tau] = \delta_{\sigma\tau}$$

and this means that the transformation from (P_ρ, Q_ρ) to (P_ρ, Q_ρ) is canonical & since the system in (P_ρ, Q_ρ) is canonical in form, the same is true of the original system.

(3) The constancy of Lagrange bracket expressions along a trajectory makes the system canonical.

From $\frac{d}{dt}[a_i, a_j] = 0$ follows

$$\frac{d}{dt}\sum_r\left(\frac{\partial q_r}{\partial a_i}\frac{\partial p_r}{\partial a_j} - \frac{\partial q_r}{\partial a_j}\frac{\partial p_r}{\partial a_i}\right) = 0.$$

ie $$\frac{d}{dt}\sum_r\sum_i\sum_j\left(\frac{\partial q_r}{\partial a_i}\frac{\partial p_r}{\partial a_j} - \frac{\partial q_r}{\partial a_j}\frac{\partial p_r}{\partial a_i}\right)\Delta a_i \delta a_j = 0$$

ie $$\frac{d}{dt}\sum_r(\Delta q_r \delta p_r - \Delta p_r \delta q_r) = 0$$

ie transformation of coordinates from t_0 to $t+dt$ is an inf. can. trn. Hence from (1)

it follows that the original system is canonical in form.

(4) If the original system becomes a canonical system under the transformation

$$p_\rho = \varphi_\rho(P_1 \cdots P_n, Q_1 \cdots Q_n)$$
$$q_\rho = \psi_\rho(P_1 \cdots P_n, Q_1 \cdots Q_n)$$

can we say anything about the transformation?

This is not in general possible for any given system. The given system must be of a special type ie

(1) It must be the Pfaffian system of a Pfaffian differential form whose bilinear covariant is of the form

$$\sum_1^{2n+1}\sum_1^{2n+1} c_{\rho\sigma}(du_\rho \delta u_\sigma - \delta u_\rho du_\sigma) \qquad \left(c_{\rho\sigma} = \frac{\partial U_\rho}{\partial u_\sigma} - \frac{\partial U_\sigma}{\partial u_\rho}\right)$$

(2) The transformation must reduce the bilinear covariant to the normal form

$$\sum\sum (du_\rho \delta u_\sigma - \delta u_\rho du_\sigma)$$

This reduction is analogous to the reduction of a general quadratic form $\sum\sum g_{\rho\sigma} du_\rho du_\sigma$ to the normal form $\sum du_\sigma^2$, and for this it is necessary that the curvature tensor should vanish. Similar result must also hold for $c_{\rho\sigma}$

II Canonical transformations.

(1) Condn for the general transformation

$$\left.\begin{aligned} p_\rho &= \varphi_\rho(P_1 \cdots P_n, Q_1 \cdots Q_n, t) \\ q_\rho &= \psi_\rho(P_1 \cdots P_n, Q_1 \cdots Q_n, t) \end{aligned}\right\}$$

being canonical.

Prange (p. 771) obtains these conditions as

$$(\varphi_\sigma, \varphi_\tau) = 0, \quad (\psi_\sigma, \psi_\tau) = 0, \quad (\varphi_\sigma, \psi_\tau) = \delta_{\sigma\tau}; \text{ and } \frac{\partial \varphi_\rho}{\partial t} = (\varphi_\rho, E), \frac{\partial \psi_\rho}{\partial t} = (\psi_\rho, E)$$

where $E = \frac{\partial \bar{W}^*}{\partial t}$ where $\bar{W}^*(q_1 \ldots q_n, Q_1 \ldots Q_n, t)$ is the substitution function given by

$\Sigma p_\rho \, dq_\rho - \Sigma P_\rho \, dQ_\rho - (H-K)\, dt = d\bar{W}^*$ ie

$$p_\rho = \frac{\partial \bar{W}^*}{\partial q_\rho}, \quad P_\rho = -\frac{\partial \bar{W}^*}{\partial Q_\rho}, \quad K = H + \frac{\partial \bar{W}^*}{\partial t}$$

ie $E = K - H =$ fn of P_ρ, Q_ρ, t.

Lectures notes on the New Hons. Syllabus (1945)

in Applied Mathematics

Kinematics: Vectors, Velocity, accelration

angular velocity - moving axes:

Lecture

Lecture notes on the new syllabus
in Astronomy – Stellar astronomy.

(1) Radiation : Vide later.

(2) 1st lecture (a) Atomic structure : atomic number; nuclear structure; atomic weight; atomic structure & periodic table (shell structure He, Ne, Ar, Kr, Xe, Rad, 2, 8, 8, 18, 18, 32) – Valence electrons eg, Na (11), Cal (20), (2 V. elec), similarity of spectra; ionisation, arc & spark spectra (ionised); normal & excited states (taking up energy & emission of energy); quanta of energy, transition, life of excited state (10^{-6} sec)

(3) (A) Spectra : collisions of 1st kind; radiation, emission & absorption spectra (monochromatic);

Relation between energy & frequency

$$\left.\begin{aligned} W_1 - W_2 &= E \\ E &= h\nu \end{aligned}\right\} \qquad h = 6.55 \times 10^{-27} \text{ unit}$$

Wave number $\frac{\nu}{c} = \eta = \frac{W_1 - W_2}{ch}$ emission or absorption acc as $W_1 \gtrless W_2$.

$\eta = \frac{W_1}{ch} - \frac{W_2}{ch}$, ie wave no = difference of two spectroscopic terms.

H-spectrum, $\eta = R\left(\frac{1}{4} - \frac{1}{m^2}\right)$ $m = 3, \ldots 29, \ldots$ Balmer lines :

ie $W_n/ch = \frac{R}{4}$, $\frac{W}{ch} = \frac{R}{m^2}$ (R = Rydberg's constant = 109,678).

Quantum conditions : (i) angle A.M $\times 2\pi \times$ angular A.M $= \tau_1 h$.

(ii) ~~$\times$~~ angular rad. moment $\times$ change of $r = \tau_2 h$. (iii) stationary states

Explanation of Balmer lines : circular orbits satisfying with $\tau_2 = 0$.

$$m\omega^2 r = Ee/r^2 \quad \text{ie } (m\omega r^2)^2 = Eerm$$

$$A.M = m\omega r^2 = \sqrt{Eerm}.$$

ie $2\pi\sqrt{Eerm} = \tau_1 h$.

$$r = \frac{\tau_1^2 h^2}{4\pi^2 mEe}$$ gives possible values of r

$$= 0.53 \times 10^{-8}\, \tau_1^2 \text{ cm}$$

$$K.E = \tfrac{1}{2} m\omega^2 r^2 = \frac{Ee}{2} = \frac{2\pi^2 m E^2 e^2}{\tau_1^2 h^2}$$

In a circular orbit $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) = \frac{\mu}{a}$

parabolic " $v^2 = \frac{2\mu}{r}$

} i.e. if K.E in circular orbit were doubled, the particle would fly off in parabolic orbit i.e. ionised.

$\therefore$ Energy photon = Energy of ionised atom − K.E. of normal atom

$$W = C - \frac{2\pi^2 m E^2 e^2}{\tau_1^2 h^2}$$

$$= C - \frac{Ch}{\tau_1^2}\left(\quad\right)$$

$$\frac{W}{Ch} = \frac{C}{Ch} - \frac{R}{K^2} \qquad (\tau_1 = K).$$

$W = T + V$

$W_i = T_i + V$

$W_i - W = T_i - T$

$= 2T - T = T$

$V = W_1 - T$

ionisation & excitation potentials; thermal ionisation, breaking up & recombination, high T favours ionisation & high ρ favours recombination, at given T & ρ ionisation $\propto \frac{1}{\text{ionisation potential}}$; enhanced lines are lines of ionised atoms in spark spectra (because ionised atom has a net +ve charge on nucleus, greater energy req'd for excitation); idea that enhanced lines are due to ionised atoms was first brought out by Saha; Saha's equation

$$\log \frac{X}{1-X} = -\frac{5048\, I}{T} + \frac{5}{2}\log T - 6.5 - \log p_e \qquad (1) \quad (\log \text{ to base } 10)$$

I = ionisation potential in volts, p_e = electron pressure like a gas, X = fraction of atoms ionised, T = abs. temp.

$$\log \frac{Y}{1-Y} = -\frac{5048\, E}{T} \qquad (2)$$

Y = fraction of neutral atoms in excited state of excitation potential E.

Band spectra:

(3) Stars: Preliminary considerations; number, constellations, designation, catalogues, charts, photographs. (B.D = Bonn Durchmusterung, 1862); (A.G = Astr. Gesellschaft). (A.C = Astrographic Catalogue)

(4) Stellar spectra:

(1) mostly absorption or dark line spectra ie atmospheres - A small proportion (<1%) show bright line spectra

(2) The line in spectra are those of known elements.

(3) Spectra can be grouped into a very small number of classes.

(4) Spectra can be arranged in a continuous sequence.

All observations photographic ; Draper classification (1918-24) (225,300 stars) gp Principal classes

O, B, A, F, G, K, M, R, N, S. & further decimal classification

Description : H lines seen throughout

O = ionised helium, He_+ , O_{++} (some stars show emission - Wolf-Rayet

B = neutral helium & ionised O_+

A = H strongest, Ca_+

F = arc lines of metals. Ca.

G (Sun) = H not prominent, arc lines of Fe , stronger than enhanced lines

K = enhanced lines weaker - TiO_2 bands appear

M = low temp. lines strong

R = can Bands of C & CN

S = " ZiO_2

By colour : O, B = bluish white (Orion)

A = white (Sirius)

F, G = yellow (Sun)

K = orange (Arcturus)

M = Red (Antares)

R, N, S = Very red.

Meaning of sequence: (1) sequence is linear (ie say from A0 → G0 one set of intermediate types)

ie one single physical condition & not chemical composition

(2) This physical condition is the degree of excitation (from ionisation theory)

(3) Differences in degree of excitation due to differences of temperatures in stellar photospheres & atmospheres.

(1) Radiation : electromagnetic types, wave length & frequency, $\nu = c/\lambda$ refractive index (ν does not change but $\lambda' = \lambda/\mu$) ; monochromatic radiation (one ν) ; flux density (intensity) ; Radiation pressure ($= I/c$) ; the spectroscope ; Doppler effect : $\lambda' = \lambda(1 + v/c)$; energy-measurement, bolometer, spectrobolometer (record energy heat metal strip & temp increases resistance & galvanometer gives indication), perfect radiator or blackbody ; Stefan's law $E = \sigma T^4$, Planck's law : $E_\lambda = \dfrac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1}$.

$E_\lambda \cdot d\lambda$ = radiant energy per second per sq cm at temp T in region λ to $\lambda + d\lambda$.

Wien's law, $\lambda_m = \dfrac{0.289}{T}$ (intensity at given T is max for λ_m) — Wien's law & Stefan's law follow from Planck's.

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(6) Stellar magnitudes (i) apparent magnitude — $\dfrac{l_m}{l_n} = 10^{-0.4(m-n)} = (2.512)^{n-m}$.

(2) Telescopic power required to show stars of given magnitude.

Given 1" aperture shows stars of 9th magnitude. | light gathering power depends
If a" " m th " | on area of object class.

$$\frac{l_m}{l_9} = \frac{i^2 a^2}{a^2} = (2.512)^{9-m} = 10^{-0.4(m-9)} \qquad (m > 9)$$

Taking logarithms (to base 10), $\dfrac{2}{5}(m - 9) = 2 \log a$

$$\text{or} \quad m = 5 \log a + 9.$$

Large telescopes because of thickness of lenses cause lot of absorption of light & this theoretical aperture is not generally accurate. Thus Yerkes 40" & Washington 26" both show nearly same magnitudes — The number of stars is however doubled.

(ii) Visual, photographic, photo-visual, bolometric magnitudes — colour index is equal to $m_{phot} - m_{vis}$. — Absolute magnitude M is magnitude at dist. of 10 parsecs $= D_0$

$$\frac{l}{L} = \frac{D_0^2}{D^2} = \frac{p^2}{p_0^2} \qquad p_0 = 10$$

$$\frac{l}{L} = 10^{0.4(M-m)} \qquad \frac{2}{5}(M - m) = \log \frac{l}{L} = 2(\log p - \log p_0)$$

ie $M - m = 5\log p - 5\log p_0$ $\left(p_0 = \frac{1}{10}\right.$

ie $M = m + 5\log p + 5$.

* Also $\frac{L_1}{L_2} = 10^{\cdot 4(M_2 - M_1)}$

For the sun, $m = -26\cdot72$, $\frac{D_0}{D} = 2,062,650$

∴ this gives $\frac{2}{5}(M-m) = 2\log\left(\frac{D_0}{D}\right)$ ie $M - m = 31\cdot57$ or $M = 4\cdot85$

∴ for any other star $\log\left(\frac{L}{\odot}\right) = \cdot4(4\cdot85 - M)$. in terms of Suns luminosity.

(7) Distributions of stars:

(i) Number of stars brighter than a given magnitude – Sears & van Rhijn's table.

(ii) Galactic concentration is the ratio of number of stars of the same magnitude in equal areas at galactic equator & galactic pole – Do table. Ratio varies from 3.4 to 44.2 from magnitude 5 to magnitude 21.

(iii) Statistical study – If a star at distance D has app. mag. m, the same star at distance $(2\cdot512\,D)$ would have magnitude $m+2$ ie Stars in the range $(m, m+2)$ would lie within spheres of ratio of volume 1 : 15.86 (or say 1 : 16). Hence if stars were uniformly distributed the ratio of increase in number for change of 2 magnitudes should be 16. But table XI shows this is not true. At galactic equator from $m = 5$ to $m = 7$, ratio is only 8 & from $m = 19$ to $m = 21$ ratio is 3.7. At galactic pole this ratio is from 7.9 to 2.2.

(iv) More than 90% stars belong to Shectral classes B, A, F, G, K & M.

) General ideas:

(a) modern ideas of cosmology – solar system – nearest star – local cluster – galaxy (Equatorial diameter 30,000 parsecs, polar diam 5000 parsecs, mass $2\times10^{11} M_\odot$) – galactic rotation, central nucleus, sun not at centre, motion of stars in orbits, period

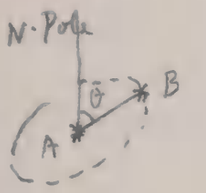
for stars near the Sun (260×10^6 years) – ~~galactic objects, individual stars, double & multiple~~ extragalactic nebulae, spiral, and etc, distance of Andromeda nebula 10^6 l.y, about same ~~as~~ as galaxy, island universes, recession of nebulae, greatest vel 25,400 m.p.s, expanding universe.

(b) Galactic objects – single stars – Proper motion – radial & tangential velocities (Doppler effect) – $T = 4.74 \mu/p$.

Solar motion & peculiar motion, solar apex & antapex – separation of solar motion – peculiar motion of stars preferential – star streaming, two-star stream & ellipsoidal hypotheses – explanation by rotation of galaxy considering motion of sun, inclination of orbit of stars, their distance from centre of galaxy which is not at the Sun.

magnitude, luminosity, surface temperature, spectral class, diameter, mass, density

Double stars – optical doubles, – physical doubles or binaries – spectroscopic binaries – eclipsing binaries – apparent orbit ~~not necessarily~~ is projection of true orbit on celestial sphere



θ = position angle, T = orbital period,

α = semi-major axis in seconds of arc, e = eccentricity.

If A = linear semi-diameter & d = distance of star

$$\sin \alpha = A/d \,, \quad \sin p = a/d \qquad (a = \text{astr. unit})$$

$$\text{or } \alpha = A/d \,, \quad p = a/d$$

$$\frac{\alpha}{p} = \frac{A}{a}$$

$$G(m_1 + m_2) = 4\pi^2 A^3/T^2 \,; \quad G(m_\odot + m') = 4\pi^2 a^3/\gamma^2 \qquad \gamma = \text{year}$$

Neglecting m' (mass of earth), taking $m_\odot = 1$, this gives, $\gamma = 1$.

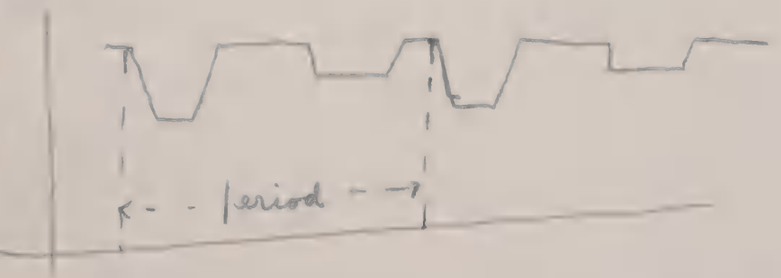
$$m_1 + m_2 = \left(\frac{\alpha}{p}\right)^3 \cdot \frac{1}{T^2}$$

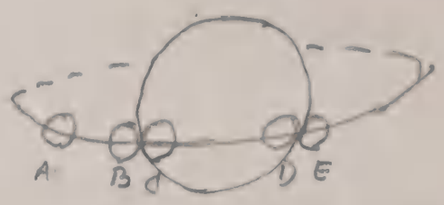
– Possibility of finding m_1 & m_2 separately – i.e sum of masses of the binary – Average mass $\frac{1}{10} m_\odot$ to $10 m_\odot$ – mass-luminosity relation due to Eddington – Dynamical parallaxes

Putting $m_1 + m_2 = 2$ gives $P = \alpha \left(\frac{1}{2T^2}\right)^{1/3}$

Eclipsing binaries – light curve is drawn for ~~A~~ apparent m (visual)

against time. There are two minima, principal & subsidiary. The light curve can be



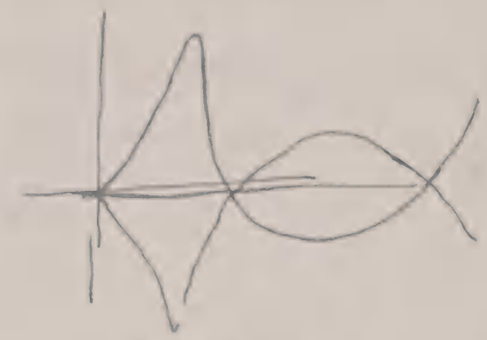


explained on the assumption of the primary being eclipsed by the secondary. Let secondary have unit luminosity per unit area & primary 100 & let surface areas be 1 & 10

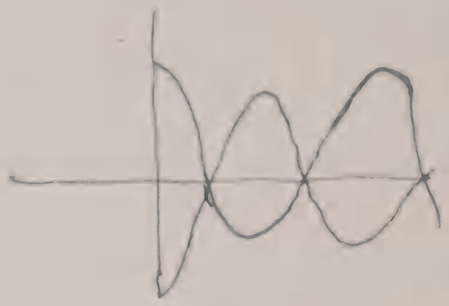
At C A luminosity = 1000 + 10 = 1010, At C it is = 900 + 10 = 910. When it goes behind luminosity = 1000. This explains the two minima

Spectroscopic binaries - radial velocity of stc observed from each spectrum variable ie spectral lines shift back & forth across their normal positions ie there is acceleration & radial vel. changes are periodic. This provides information of the relative motion of components - Velocity curves

one spectrum



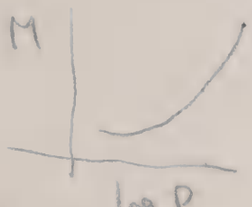
both spectra



both spectra

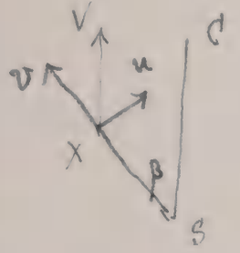
The velocity curves determine orbits & give data to find m_1 & m_2 also separately.

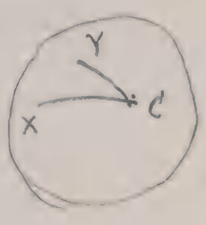
Variable stars - Periodic variables - long period - short period or cepheid variables (few hrs to few days) - Period luminosity relation for them & its significance to find stellar parallax - irregular variables - Novae - Cepheids outside galactic system also.



Star clusters - Open & globular - globular clusters at boundary of Galaxy

moving clusters - proper motions converge towards a point or diverge from one. Let V be velocity of one star X of the cluster & SC parallel to V thus' S (Sun). u & v are transverse & radial velocities. C is the sign of the convergent point. on the celestial sphere proper motions of stars X & Y of the cluster are directed along CX & CY. If C be known, CX = β is known & v (rad. vel) = V cos β is known and thus therefore determines V. For any other star u = V sin β = 4·74μ/p and knowing β & μ, the knowledge of the common V for the cluster determines p for the star & hence its distance.





Nebulae - Planetary nebulae - diffuse nebulae - Dark nebulae.

9) Luminosity, temperature and diameter - Giant & dwarf stars definition by luminosity or M (-5 to 2) & (2 to 15) - Super giants - white dwarfs - ~~Spectra of giants & dwarfs~~ Relation between M, T and R -

From Planck's law of radiation, the amount of radiation between wavelengths λ and λ+dλ emitted per unit area per second is given by

$$J_\lambda = C_1 \lambda^{-5} / (e^{C_2/\lambda T} - 1) \quad \text{or} = \frac{C\lambda^{-5}}{(10^{K/\lambda T} - 1)}$$ C & K being constants

$$\therefore L_\lambda = 4\pi R^2 J_\lambda$$

$$M_\lambda = 2.5(\log L_0 - \log L_\lambda) \qquad [\text{taking } M_0 = 0,\ L_0 \text{ a const}$$

$$\therefore M_\lambda = 2.5 \log (L_0/4\pi) - 5 \log R - 2.5 \log J_\lambda$$

$$10^{K/\lambda T} - 1 = 10^{K/\lambda T}(1 - 10^{-K/\lambda T})$$

$$\& \log J_\lambda = \log (C\lambda^{-5}) - K/\lambda T - \log (1 - 10^{-K/\lambda T})$$

$$\therefore M_\lambda = 2.5 \log\left(L_0 \lambda^5/4\pi c\right) - 5\log R + 2.5^- K/\lambda T + 2.5 \log\left(1 - 10^{-K/\lambda T}\right)$$

which can be written in the form

$$M_\lambda = C_\lambda - 5\log R + 1.56/\lambda T + x \qquad (1) \qquad [K = .624$$

x being the last term which is small & is a function of $K/\lambda T$ only.

Substituting standard values for wave length effective for visual & photographic estimates

$$\left.\begin{aligned} M_v &= C_v - 5\log R + \frac{29{,}500}{T} + x_v \\ M_p &= C_p - 5\log R + \frac{36{,}700}{T} + x_p \end{aligned}\right\} \quad (2), (3)$$

For the sun, $R_\odot = 1$, $T = 6000^\circ$, $x = -0.01$, $M_v = 4.83$ giving $C_v = -0.08$

and with $M_p = 5.40$, $C_p = -0.72$

x_p & x_v are smaller than the error in determinations of M_p & M_v & can be neglected.

Thus
$$\left.\begin{aligned} M_v &= \frac{29500}{T} - 5\log R - 0.08 \\ M_p &= \frac{36700}{T} - 5\log R - 0.72 \end{aligned}\right\} \quad (4)$$

giving relation between luminosity, surface temperature & radius of the star.

The colour-index $I = M_p - M_v = \frac{7200}{T} - 0.64$ (5)

depends only T and $T = \frac{7200}{I + 0.64}$. (5')

For bolometric magnitudes M_τ we use Stefan's law

$$E = \sigma T^4$$

and get $M_\tau = C_\tau - 5\log R - 10\log T$

& for the Sun $C_\tau = 42.0$, so that

$$M_\tau = 42.0 - 5\log R - 10\log T \qquad (6)$$

and the heat index $= M_\tau - M_v = 10\log T + \frac{29500}{T} - 42.1$ (7)

The heat index is not of much practical importance because earth's atmosphere is opaque to radiation of $\lambda < \lambda 3000$ A.U. From (7) it follows that the heat index is a minimum for $T = 7000°$.

From (2) and (5') we can eliminate T and get

$$\log R = 0.82\, I - 0.20\, M_v + 0.51 \qquad (8)$$

which gives the radius of a star knowing M_v and I.

Let d'' be the diameter of the star in angular measure (~~seconds of arc~~) and p'' its parallax. If D be the distance of the star in parsecs

$2R = D \cdot d$ (d in radian measure)

$D = 1/p$ (p in radian measure & D in astr. units

~~$\therefore d = 2R/p$~~ (~~p in seconds & R in parsecs~~)

~~In (8) R is in terms of $R_\odot$~~

$$\therefore d = 2Rp \quad (R \text{ in astr. units})$$

$$\therefore d'' = 2Rp''$$

In (8) R is in interms of $R_\odot$ & $R_\odot = \frac{1}{107}$ astr. unit

$$\therefore d'' = \frac{pR}{107} \qquad (9)$$

$$\therefore \log d'' = \log p + \log R - 3.05 \qquad (10)$$

Also $M_v = m_v + 5 + 5 \log p$ (~~6~~) (11)

From (8), (10) and (11) we have, eliminating ~~log~~ $\log p$ and $\log R$

$$\log d'' = 0.82\, I - 0.20\, m_v - 2.52 \qquad (12)$$

$$\text{or } \log d'' = \frac{5900}{T} - 0.20\, m_v - 3.05 \qquad (13)$$

since M_v falls out in course of elimination.

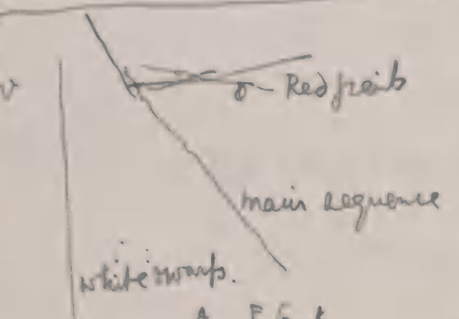
(13) gives T if d'' & m_v be known by observation. d'' can be found for the largest stars by method of stellar interferometer.

For stars of small diameter (R) the stellar interferometer is not available, for eg the white dwarfs. For these Einstein's theory of relativity is applied. According to this theory light waves in a strong gravitational field (for such field potential $\propto \mu/R$ & small R since strong fields) suffer diminution in ν or there is a shift to the red. This red-shift can be determined. For eg. companion of Sirius we get $R = 3\,R_{earth}$ while $m = \cdot 96\,M_\odot$ (m = mass) Thus we get density = 50,000 times that of water - dimensions of a planet but mass of the sun.

(10) The mass-luminosity relation : $L \propto M^{4+\beta}$ $+^{-1/2}$ - majority of stars except white dwarfs.

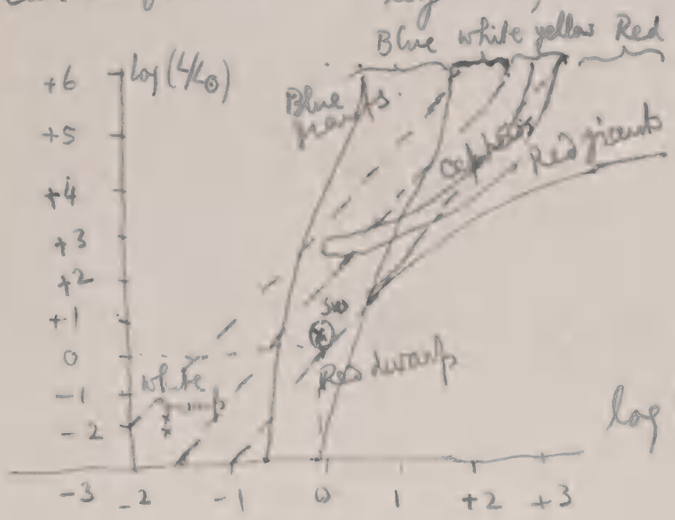
(11) Hertzsprung-Russell diagram. Relation between absolute magnitude & spectral

Division into main sequence, giants, white dwarfs.



Explanation of H-R diagram ie explanation of relationship between luminosity and surface temperature as also explanation of mass-luminosity relation is a problem of internal constitution of stars.

The Hertzsprung-Russell or R-H-D can also be written by plotting L and R & is equivalent to the R-H-D, for spectral class is an estimate of T and given L & T, R can be found. Plotting $\log(L/L_\odot)$ against $\log(R/R_\odot)$ gives the R-L diagram (R.H.D is also a L-T-D)



R-L diagram.

main sequence Stars can be characterised by two parameters L and T & a curve in the diagram means a one-parameter group.

L & R increase regularly with mass & so T

Main-sequence - one-parameter group from blue giants to red dwarfs, the former hot & luminous & latter cool & faint

Red giants (L large, T small) with L and R larger than for stars of the same mass in main sequence. These form giant branch branching off near the middle of main sequence. The upper boundary of this branch

Consist of Cepheid variables.

White dwarfs – left & below main sequence (L small, T large) & smaller L & R than Corresp stars of corresponding mass in the main sequence. Relates to these are Wolf-Rayet stars (nucleii of Planetary nebulae).

That the explanation of the mass-luminosity relation and the R-H-D is a problem of the interior of a star is evident from the fact that the luminosity of a star is due to the radiation energy flowing from the interior to the surface, and also there should be a suitable distribution of temperature to make this flow possible & the surface value of this T is the effective temperature.

~~Hoyle, Russell Theorem – the state of a star is completely specified by two parameters and hence completely specified by its position in the R-H-D.~~

Elementary method of procedure for the theory of internal constitution of a star.

methods possible because matter in gaseous & ionized states at high temps & pressures have simpler properties

(a) mechanical equilibrium – P (Pressure) can be calculated if law of variation of density ρ with $\frac{1}{r}$ be known ie model be known.

(b) Validity of eqn of state of perfect gas – for given model T can be calculated since μ (mean molecular wt) can be found from atomic theory.

(c) Radiation pressure – taking this also account since it $\propto T^4$, T at any point can again be found.

Eg: Eddington's model (P_c = 2.8 that of world), $P_c = 36 \times 10^9$ atm, $T_c = 2.9 \times 10^7$ for ⊙.

(d) Transition from interior to surface — Transport of radiation & resistance due to opacity. The coeff of opacity $\chi = \chi(P, \mu, T)$ from quantum theory. Eqn of radiative eqm gives an integration L & T at surface.

The result of this investigation is that $L \propto M^4 r^{-1/2}$ for Eddington's model, and change of model also makes little difference. dependence on r also is slight ($r^{-1/2}$). Chemical composition (except for abundance of H) also makes little difference. Thus that L mainly depends on M is the meaning of Eddington's mass-luminosity relation.

Alternatively, the above investigation also means that from the given the mass, radius of the model, L and T can be calculated from pure theory. In this another important variable factor viz chemical composition is not taken into consideration ie no account is taken of energy generation in the interior which is fundamental. If this be also taken into consideration it can be shown that for given mass & chemical composition (mainly abundance of H) radiative eqm is possible for a given model & radius ie for given mass and composition, eqm is possible for only one value of the radius and hence for definite values of L and T. This gives Vogt-Russell theorem viz that the state of a star is characterised by two parameters. If chemical composition be fixed, relation of L and T is only on mass ie we have a one-parameter family of solutions ie a line in the R-H-D. ~~The mass-luminosity~~ The matter in a star is specified by its mass & chemical composition. If the latter be same in the original condition of all stars, change in chemical composition can only be due to energy-generating processes ie change due to age. Thus mass and age can be taken as the two parameters. The mass-luminosity relation is obtained as a relation between two parameters, the stars of different luminosities in the main sequence being also of different masses ie the mass-luminosity relation merely describes uniform chemical composition.

Thus interpretation of R-H-D is 2-parameter problem requires study of energy-generation.

(13) More detailed study of stellar ~~mechanical~~ equilibrium.

(a) Mechanical equilibrium. (a) Mechanical equilibrium — The inward pull of gravitation must be balanced by the net outward pressure of gas

Homologous stars are those built on the same 'model' ie same law of density. Corresponding points are those at which (r/R) is same. At corresp. points (ρ/ρ_m) is the same (ρ_m = mean density).

Lane's laws: At corresponding points $\rho \propto M/r^3$, $P \propto M^2/r^4$, $T \propto \frac{mM}{r}$ (m = mol. wt)

Proof: Since ρ/ρ_m = same at corresp. points

$$\frac{\rho_1}{\rho_2} = \frac{M_1/r_1^3}{M_2/r_2^3}$$

Pressure is given by gravitational force/unit area & gravitational force $\propto G\frac{mm'}{d^2}$

$$\therefore \quad \frac{P_1}{P_2} = \frac{M_1^2/r_1^4}{M_2^2/r_2^4}.$$

From the perfect gas law assumed valid, $p = \frac{R}{m}\rho T$

and taking gas pressure = gravitational pressures $\frac{T_1}{T_2} = \frac{m_1 M_1/r_1}{m_2 M_2/r_2}$.

Consider first a sphere of constant ρ_0, mass M & radius R

$$M = \tfrac{4}{3}\pi\rho_0 r^3 \quad \text{ie} \quad \rho_0 = \frac{3}{4\pi}\frac{M}{r^3}.$$

Surface gravity $= g_1 = GM/r^2$. pressure P_0 at centre = wt of ~~mist~~ column of unit cross section of height r and density $\rho_0 = \frac{1}{2}g_1\rho_0 r$ (here $g \propto r$ & hence average value $= \frac{1}{2}g_1$)

$$P_0 = \tfrac{1}{2}\rho_0 g_1 r = \frac{3G}{8\pi}\frac{M^2}{r^4}$$

For the temperature T, with $\mathcal{R}P = p$

$$T_0 = \frac{m}{R}\frac{P_0}{\rho_0} = \frac{G}{2R}\frac{mM}{r}$$

These eqns satisfy Lane's laws obviously. To apply them to any star, we can write its density at any point $\rho = X\rho_0$ with ρ_0 the mean density having the above value. X is known when law of density is known. Similarly let $P = \gamma P_0$ ie γ means what the central pressure would be if the star were homogeneous. Then $T = \frac{m}{R}\frac{P}{\rho} = \frac{m}{R}\cdot\frac{P_0}{\rho_0}\cdot\frac{\gamma}{X}$

Thus $\rho = \frac{3}{4\pi}\frac{M}{r^3}X$; $P = \frac{3G}{8\pi}\frac{M^2}{r^4}Y$, $T = T_0\frac{Y}{X} = \frac{G}{2R}\frac{mM}{r}\frac{Y}{X}$ (1)

(c) Taking radiation pressure into consideration, if the total gas+radiation pressure be P, let the gas pressure balance βP and the radiation pressure q the balance

$$\text{ie } p = \beta P, \quad q = (1-\beta)P.$$

Lane's law remain the same with the factor β appearing. Thus

$$\frac{T_1}{T_2} = \frac{\beta_1 m_1 M_1/r_1}{\beta_2 m_2 M_2/r_2}.$$

For the radiation pressure $q = \sigma T^4$ & this is as unfortunate as P_0 at high T.

Now, from (1), $p = \frac{R}{m}\rho T = \frac{3R}{4m}\frac{M}{r^3}XT$

$$P = \frac{3G}{8\pi}\frac{M^2}{r^4}Y$$

Hence $\beta = p/P = \frac{2R}{G}\cdot\frac{r}{mM}\frac{X}{Y}T$ (2)

Also $1-\beta = q/P = \frac{8\pi\sigma}{9G}\frac{r^4}{M^2}\frac{T^4}{Y}$ (3)

Eliminating T between (2) & (3) gives

$$\frac{1-\beta}{\beta^4} = \frac{\pi}{18}\frac{\sigma G^3}{R^4}\frac{Y^3}{X^4}m^4M^2 \qquad (4)$$

If the model be known, X and Y are known, and β can be found. (4) shows that β depends not on r but only m, M & density law. Having found β, eqn (2) gives the temperature distribution. The numerical coefficient in (4) can be removed by expressing M in terms of $M_\odot = 1.99\times10^{33}$ gms, $M_\odot^2 = 3.96\times10^{66}$. The numerical coefficient $= 8.21\times10^{-69}$ and (4) becomes

$$\frac{1-\beta}{\beta^4} = 0.0325\frac{Y^3}{X^4}m^4\left(\frac{M}{\odot}\right)^2 \qquad (5)$$

which is Eddington's equation.

For Eddington's model, m is taken constant, and r^3/x^4 = const ie pressure $\propto \rho^{4/3}$. This by use of gravitational & Gas laws gives x and γ and in particular at the centre $x = 54$, $\gamma = 91$. (5) then becomes

$$\frac{1-\beta}{\beta^4} = 0.0031\, m^4 \left(M/M_\odot\right)^2 \qquad (6)$$

From atomic consistencies relating to abundance of light & heavy atoms m can be put = 2.11 4

(6) becomes

$$\frac{1-\beta}{\beta^4} = 0.061 \left(M/M_\odot\right)^2 \qquad (7)$$

For this model therefore we can find P_c, P_c and T_c ($P_c = 36 \times 10^9$ atm, $T_c = 29 \times 10^6$ K).

c) Radiative equilibrium. This depends on the opacity which is given for a layer of gas of thickness h and density ρ by $\kappa\rho h$ where κ = opacity coefficient given by

$$\kappa = \frac{K\rho}{m T^{7/2}} \qquad (8)$$

Rate of flow of heat per unit area through a layer $(r, r+dr)$ denoted by H is $\propto \frac{1}{\text{opacity}}$ of layer, and is equal to decrease of radiation pressure dq multiplied by $\underline{c}$ (velocity of light) ie

$$H = \frac{c}{K\rho}\frac{dq}{dr} \qquad (9)$$

this differential eqn of radiative equilibrium.

From mechanical equilibrium

$$\frac{dP}{dr} = \rho g \qquad (g = \text{accn of gravity}) \qquad (10)$$

this Hydrostatic eqn

Hence from (9) and (10), $\frac{H}{g} = \frac{c}{K}\frac{dq}{dP}$ (11)

If E = amount of heat escaping thro' layer per unit time, $H = E/4\pi r^2$

and $g = GM_r/r^2$.

$$\therefore \quad \frac{E}{M_r} = \frac{4\pi c G}{K} \frac{dq}{dP} \qquad \text{and } K = \frac{kP}{m T^{7/2}}$$

$$\therefore \quad \frac{E}{M_r} = \frac{4\pi c G}{K} \frac{m T^{7/2}}{P} \frac{dq}{dP} \qquad (12)$$

With $q = (1-\beta) P$, this differential equation can be transformed & could be integrated with approximately. In the case of Eddington's model, β is constant throughout the star, and taking the layer at the photosphere (12) gives

$$E = \frac{4\pi c G}{K} \frac{m T^{7/2}}{P} M(1-\beta) \qquad (13)$$

Using Lane's laws ie from (1) for ρ and for T, using (2) since rad^{n} pressure is also to be considered

$$\frac{T^{7/2}}{\rho} = \frac{4\pi}{3} \left(\frac{G}{2K}\right)^{7/2} \cdot \frac{\gamma^{7/2}}{\chi^{9/2}} \cdot \frac{M^{5/2}}{r^{11/2}} (\beta m)^{7/2}$$

From (4), we can express β^4 in terms of $(1-\beta)$ & hence also $\beta^{7/2}$ in terms of $(1-\beta)^{7/8}$ and substituting for $T^{7/2}/\rho$ in (13) gives, replacing E by L

$$L = C M^{7/4} (1-\beta)^{15/8} r^{-1/2} m. \qquad (14)$$

~~With m = const for Eddington's model~~

$$L \simeq C M^{7/4} (1-\beta)^{15/8} r^{-1/2} \qquad (14)$$

which is the statement Eddington's mass - luminosity relation.

Introducing effective temperature T in place of r, we use

$$L = 4\pi r^2 \sigma T^4 \qquad (15)$$

Eliminating r from (14) & (15) gives

$$L = C' M^{7/5} (1-\beta)^{3/2} m^{4/5} T^{4/5} \qquad (16)$$

Instead of the luminosity L can be introduced the bolometric $M_{bol} = \frac{5}{2}(\log E_0 - \log E)$

giving the equation

M_b M_g = const − 3.75 log (1−β) − 3.5 log M − 2 log T. (17)

giving relation of M_b on M and T.

on the R-H-D a *line* means a rela like $f(E,T)=0$ or $f(M_b,T)=0$ ie hence a relation like $f(M,E)$ or $f(M,L)=0$ which is a statement of mass-lum. rel^{n}.

(14) *Energy-generation* :

4 Possible sources – nuclear energy given out in nuclear reactions – at surface atomic structure, interior nuclear structure – ~~Keesmus~~-nuclear reactions to be considered – Bethes analysis showing that reactions between H & light nuclei are the sources – In general terms energy production is due entirely to formation of 4 atoms of Hydrogen into one of He, resulting in a diminution of combined masses by 1 part in 135 – Bethes results

(i) no elements heavier than He can be permanently built under conditions in stellar interiors

(ii) Thermonuclear reactions cannot take place (at stellar interior temps) with elements heavier than N.

There leave only possibilities (divided into three classes)

(1) $H^1 + H^1 = D + e^+$ 4 D → He^4.

~~(2) $H^1 + Li^7$; $H^1 + Be^9$; $H^1 + B^{11}$~~

(2) $H^1 + Li^7 = 2\,He^4$

$H^1 + Be^9 = Li^6 + He^4$

$H^1 + B^{11} = 3\,He^4$

(3) $H^1 + N^{14} = O^{15}$

class (1) life time 10^{11} years – energy generation in erg/g.sec is 0·2 – T_c for producing reaction in 10^6 ·4

(2) ⊙ .. 10^3 (burn away in "some") – ~ 10^{10} – 2 to 5

(3) 10^6 or 10^7 – 3 – 15 to 19.

Also in (2) Li, Be, B are not replaced etc for eg the chain reaction for Be^9 is

$$Be^9 + H^1 = Li^6 + He^4$$
$$Li^6 + H^1 = Be^7$$
$$Be^7 + e^- = Li^7$$
$$Li^7 + H^1 = 2He^4$$

ie Be is not replaced & end product is He^4.

	Z	A
H	1	1
He	2	4
Li	3	6.94
Be	4	9.02
B	5	10.85
C	6	12.00
N	7	14.01
O	8	16.000

In (3) written as a chain, $N^{14} + H^1 = O^{15}$ is the middle of the chain

$$C^{12} + H^1 = N^{13}$$
$$N^{13} = C^{13} + e^+$$
$$C^{13} + H^1 = N^{14}$$
$$N^{14} + H^1 = O^{15}$$
$$O^{15} \text{ etc} = N^{15} + e^+$$
$$N^{15} + H^1 = C^{12} + He^4$$

ie the C^{12} is again obtained - This is called the C-N-cycle - most impt. source of stellar energy - C & N are merely catalysts - life time of $C^{12} + H^1$ is 2.5×10^6 years & $N^{14} + H^1$ is 5×10^7 yrs ie times short of 2×10^9 years ie number of cycles have taken place & steady state established.

Common to (1) (2) (3) is end product He^4 ie α-particle. Nothing can happen to it $\because He^4 + H = Li^5$ which is unstable since Li^5 does not exist. Another characteristic of (3) is it $\propto T^{18}$ ie highly susceptible to change of T_c.

Sun - only (3) gives right T_c as calculated by Eddington.

Main sequence: From known surface energy rate of energy generation, T_c can be calculated from nuclear data assuming H-content 35% & N-content 10% for main sequence stars. These coincide with T_c obtained by Eddington's method - T_c almost same although L varies by 10^6. This is because of T^{18} - narrow width of main sequence is due to the limitations (a) not so young that generation is due to contraction or burning of light elements

(b) must not fail to possess H. || in main sequence H-content could be

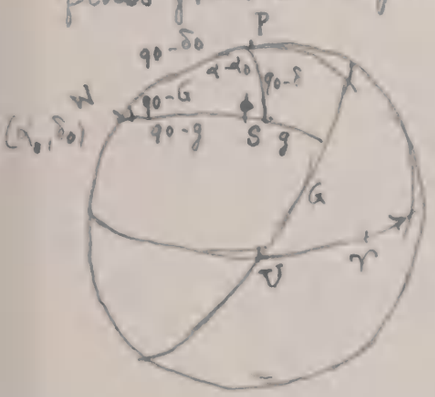
Giants satisfy (a), white dwarfs (b). ; from wafer-

Giants · $T_C = \frac{1}{10} T_C$ for main sequence - Carbon cycle does not work, Li, Be, B reactions take place & also contraction - giants are young - Li ch scarce on Earth - Transition curves for Li, Be & then B and then contracts & on to main sequence - Between Li reach & contract limit is line of Cepheids.

White dwarfs & novae - Chandrasekhar's recalls $1.4 M_\odot$, M_0 & $> M_0$ - Complete degeneracy, degenerate core, no-degeneracy -

Pictures of evolution - Small masses & big masses.

5) Stellar Dynamics: From counts of stars it is possible to define a plane of maximum stellar distribution, which defines determines the galactic plane. The coordinates of the pole of this great circle is given by $\alpha = 190^\circ, \delta = +28^\circ$ (1900).



Relation between galactic coords & equatorial coordinates can be easily found. g is counted positive if S be in the hemisphere containing P. From the Δ PWG

$$\left.\begin{aligned} \sin(\alpha - \alpha_0)\tan G &= \cos\delta_0 \tan\delta - \sin\delta_0 \cos(\alpha - \alpha_0) \\ \sin g &= \sin\delta_0 \sin\delta + \cos\delta_0 \cos\delta \cos(\alpha - \alpha_0) \end{aligned}\right\} \quad (1)$$

Also $P\hat{S}W$ called the galactic parallactic angle is given by

$$\sin(\alpha - \alpha_0)\cot\phi = \cos\delta \tan\delta_0 - \sin\delta \cos(\alpha - \alpha_0) \quad (2)$$

Components of proper motion in galactic coordinates: If the arc SS' on the celestial sphere denotes the proper motion μ per year, and θ the position angle (hence $P\hat{S}S'$ (measured at S eastwards from meridian SP)

$$\mu_\alpha \cos\delta = \mu \sin\theta \; ; \; \mu_\delta = \mu\cos\theta \quad (3)$$

For the transverse velocity

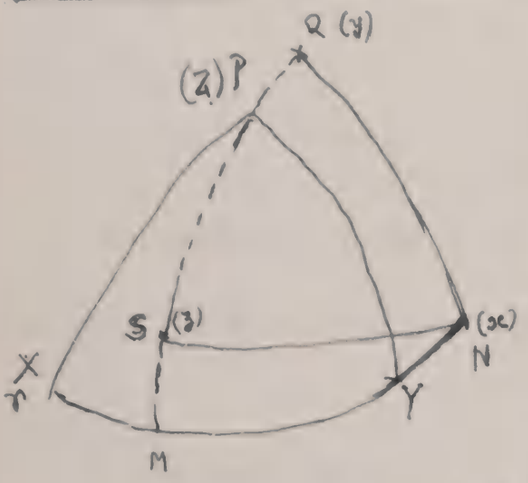
$$T = \frac{\eta\mu}{p} \quad (\eta = 4.74) \text{ in km/sec.} \quad (4)$$

The components of proper motion μ_G & μ_g are given by (analogous to (3))

$$\mu_G \cos g = \mu \sin WSS' \; ; \; \mu_g = \mu\cos WSS'$$

$$\text{ie } \mu_G \cos g = \mu \sin(\phi + \theta) = \mu \sin\phi \cos\theta + \mu \cos\phi \sin\theta$$
$$= \mu_\eta = \mu_\alpha \cos\delta \cos\phi + \mu_\delta \sin\phi$$
$$\text{Simly. } \mu_g = -\mu_\alpha \cos\delta \sin\phi + \mu_\delta \cos\phi \qquad (5)$$

Linear components of stellar velocity. If S be the star,



$\mu_\alpha \cos\delta$ which is $\parallel^l$ to the equator is $\parallel^l$ to ON ($MN = 9$
μ_δ " along SP " OQ ($SQ = 90$

the radial velocity R is along OS.

The linear components of velocities are $\frac{\kappa \mu_\alpha \cos\delta}{p}$, $\frac{\kappa \mu_\delta}{p}$ and

$\therefore$ If u, v, w the velocity components in the equatorial system

$$u = \frac{\kappa \mu_\alpha \cos\delta}{p} \cos NX + \frac{\kappa \mu_\delta}{p} \cos QX + R \cos SX$$
$$v = \text{"} \quad \cos NY + \text{"} \quad \cos QY + \text{"} \quad \cos SY$$
$$w = \text{"} \quad \cos NZ + \text{"} \quad \cos QZ + \text{"} \quad \cos SZ$$

The equatorial coordinates of X, Y, Z are $(0,0)$, $(90^\circ, 0)$, $(0, 90^\circ)$

and of N, Q, S are $(90^\circ + \alpha, 0)$, $(\alpha, 90^\circ + \delta)$, $(\alpha, +\delta)$

so that we could write $\cos NX, \ldots ; \cos QX, \ldots ; \cos SX, \ldots$

using $\cos S_1 S_2 = \sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(\alpha_1 - \alpha_2)$.

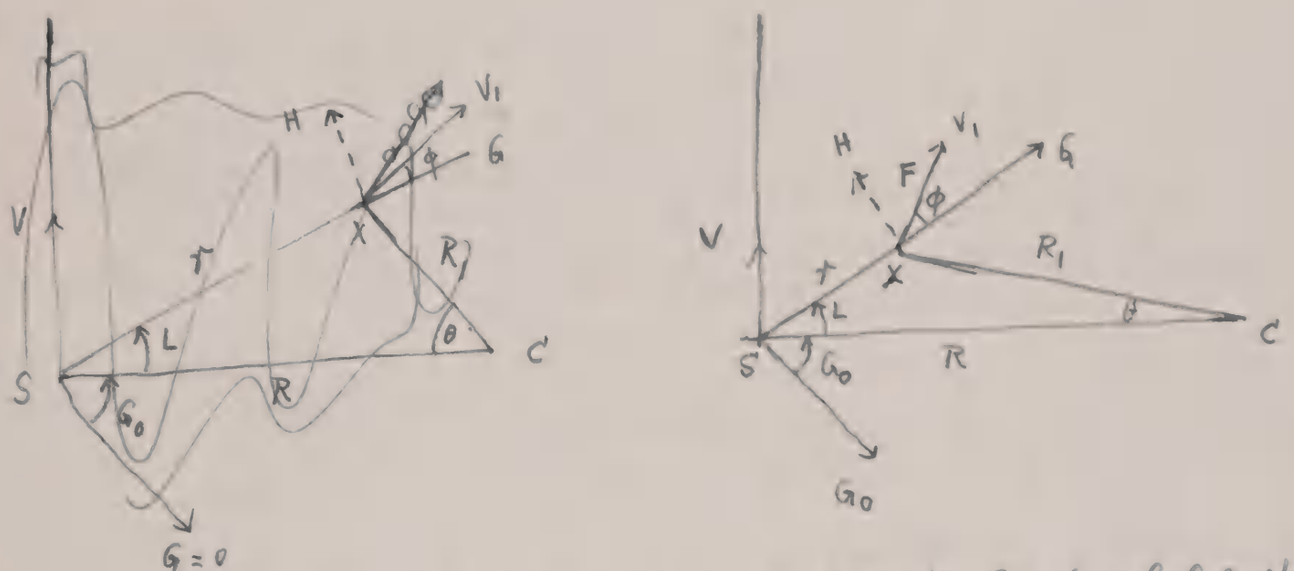
Thereforie
$$u = -\kappa \frac{\mu_\alpha}{p} \sin\alpha \cos\delta - \kappa \frac{\mu_\delta}{p} \cos\alpha \sin\delta + R \cos\alpha \cos\delta$$
$$v = \kappa \frac{\mu_\alpha}{p} \cos\alpha \cos\delta - \kappa \frac{\mu_\delta}{p} \sin\alpha \sin\delta + R \sin\alpha \cos\delta$$
$$w = \kappa \frac{\mu_\delta}{p} \cos\delta + R \sin\delta \qquad (6)$$

We can also find the linear components of the velocity in galactic coordinates.

Denoting these by ξ, η, ζ we can write expressions for these

exactly as in (6) replacing u, v, w by ξ, η, ζ and (α, δ) by (G, g) and (μ_α, μ_δ) by (μ_G, μ_g). We can then substitute for μ_G and μ_g in terms of μ_α and μ_δ from (5), and finally replace (ϑ, G, ϕ) by $(\alpha, \delta, \alpha_0, \delta_0)$ using the transformation between galactic & equatorial coordinates.

Galactic rotation : Assuming this rotation we shall derive Oort's formulae for radial velocity and proper motion for stars in the neighbourhood of the Sun. Shapley's work has shown that the galactic centre is in the galactic plane in the direction of longitude 325° (Galactic ... also observational data regarding shows that the rotation is in the retrograde direction i.e. opposite to the direction in which galactic longitude is measured.



Consider the star X in the galactic plane ($g = 0$) and S is the Sun (or C.G of stars close to S) + small compared to R (i.e. $(r/R)^2$ etc neglected). C is the galactic centre. Velocities of S & X ⊥r to CS & CX are V & V_1 in senses shown i.e. clockwise rotation (of the Galaxy) in direction opposite to measurement of G which is contra-clockwise. Plane of paper is galactic plane

$$L = G - G_0 \qquad (1)$$

Let K be gravitational attractive force per unit mass at S & $K = K(R)$.

$$\frac{V^2}{R} = K(R) \qquad (2)$$

&

$$\frac{V_1^2}{R_1^2} = K(R_1) \qquad (3)$$

The radial velocity ρ of X, which is velocity of X in direction SX relative to S is

$$\rho = V_1 \cos\phi - V \sin L$$

and $\phi = 90 - L - \theta$

$$\therefore \quad \rho = V_1 \sin L \cos\theta + V_1 \cos L \sin\theta - V \sin L \tag{4}$$

Also $R_1 = R + \Delta R$ gives from

$$R_1^2 = R^2 - 2Rr\cos L + r^2$$

$$R^2 + 2R\Delta R = R^2\left(1 - \frac{2r}{R}\cos L\right) \quad \text{neglecting } (\Delta R)^2 \text{ & } (r/R)^2$$

$$\text{ie} \qquad \Delta R = -r\cos L \tag{5}$$

V_1 is a fn of R_1 ie $V_1 = f(R_1) = f(R + \Delta R) = f(R) + \Delta R \frac{df}{dR}$ & $V = f(R)$

$$\therefore \quad V_1 = V - r\frac{\partial V}{\partial R}\cos L \tag{6}$$

Also $\dfrac{\sin\theta}{r} = \dfrac{\sin L}{R_1}$ or $\sin\theta = \dfrac{r}{R}\sin L$, to the first approximation

ie $\cos\theta$ can be put $= 1$.

Hence (4) gives $$\rho = \left(V - r\frac{dV}{dR}\cos L\right)\left(\sin L + \frac{r}{R}\cos L \sin L\right) - V\sin L$$

$$\rho = r A \sin 2L \tag{7}$$

the term $\left(-r\frac{dV}{dR}\cos L\right)\left(\frac{r}{R}\cos L \sin L\right)$ being neglected since $-r\cos L = \Delta R$ & we have $(\Delta R)\left(\frac{r}{R}\right)$ an infinitesimal of 2nd order. In (7)

$$A = \frac{1}{2}\frac{V}{R} - \frac{1}{2}\frac{dV}{dR} \tag{8}$$

Using (1), $$\rho = r A \sin 2(G - G_0) \tag{9}$$

That $\rho \propto r$ is verified by observation

If $g \neq 0$ ie star in galactic latitude g, (9) can be generalised & we get

$$\rho = r A \sin 2(G - G_0)\cos^2 g \tag{10}$$

As regards proper motion, we only have (since $g = 0$) one component of transverse velocity (T) viz since T is measured relative to S

$$T = V_1 \sin\phi - V\cos L$$

$$= \theta\left(V - r\frac{dV}{dR}\cos L\right)\left(\cos L - \sin L \sin\theta\right) - V\cos L$$

$$= -r\frac{V}{R}\sin^2 L - r\frac{dV}{dR}\cos^2 L$$

$$= r\left\{\frac{1}{2}\frac{V}{R} - \frac{1}{2}\frac{dV}{dR}\right\}\cos 2L - r\left\{\frac{V}{2R} + \frac{1}{2}\frac{dV}{dR}\right\} \qquad (11)$$

or $$T = rA\cos 2(G - G_0) + rB \qquad (12)$$

where $$B = A - \frac{V}{R}$$

From $T = \mu/p$; $\mu = \frac{T}{r}$ $\Rightarrow$ $$\mu_G = A\cos 2(G - G_0) + B \qquad (13)$$

which is also independent of r.

For the general case $g \neq 0$, we have also μ_g besides μ_G and it can be shown, calling $\mu'_G = \mu_G \cos g$ and introducing K that in this case

$$K\mu'_G = A\cos 2(G - G_0)\cos g + B\cos g \qquad (14)$$

and $$K\mu_g = -\frac{1}{2}A\sin 2(G - G_0)\sin 2g \qquad (15)$$

As seen from (13), the mean value of μ_G for $0 \leq G - G_0 \leq 2\pi$ is B the periodic fn having mean value 0. Thus the sign of B is the same as that of the mean value of μ_G & hence determines the sign of galactic rotation. In the figure $\mu_G < T$ are in direction of increasing G ie direct. If proper motion be due to rotation of galaxy it should be retrograde ie B should be negative, as has actually been determined experimentally by observation

Oort's constants A and B : In the above these are defined by V & R ie relate to the Sun and their determination depends upon the law $K(R)$.

We can express A and B in terms of K.

$$V^2/R = K \qquad (1)$$

$$\therefore 2V\frac{dV}{dR}\cdot\frac{1}{R} - \frac{V^2}{R^2} = \frac{dK}{dR}$$

$$\text{ie } 2V\frac{dV}{dR} = \frac{V^2}{R} + R\frac{dK}{dR} = K + R\frac{dK}{dR}$$

$$\text{ie } V\frac{dV}{dR} = \frac{1}{2}K + \frac{1}{2}R\frac{dK}{dR} \qquad (2)$$

$$A = \frac{1}{2}\frac{V}{R} - \frac{1}{2}\frac{dV}{dR} = \frac{1}{4}\frac{V}{R}\left(1 - \frac{R}{K}\frac{dK}{dR}\right)$$

$\frac{V}{R}$ = angular velocity of rotation at $S = \omega$, say, $= \omega_0$

$$A = \frac{1}{4}\omega\left(1 - \frac{R}{K}\frac{dK}{dR}\right) \qquad (3)$$

and $$B = A - \frac{V}{R} = A - \omega \qquad (4)$$

For the dependence of K on R, we consider two cases (A use suffixes 1 and 2 for these cases)

(1) assuming a mass concentrated at the centre of galaxy

(2) treating galaxy like a solid body

ie (1) $K_1 = \frac{C}{R^2}$ (2) $K_2 = DR$, assumptions respectively of Oort & Lindblad.

C & D being constants.

(1) $$V_1^2 = K_1 R = \frac{C}{R}$$

$$\therefore A_1 = \frac{1}{4}\frac{V_1}{R}\left(1 - \frac{R}{K_1}\frac{dK_1}{dR}\right) = \frac{1}{4}\frac{V_1}{R}\left(1 + \frac{R^3}{C}\cdot\frac{2C}{R^3}\right) = \frac{3}{4}\omega_1 \qquad (5)$$

$$B_1 = A_1 - \omega_1 = -\frac{1}{4}\omega_1 \qquad (6)$$

(2) Here $A_2 = 0$, $B_2 = -\frac{V_2}{R} = -\omega_2 = -D^{1/2}$

In this case there is no differential effect ie no dependence on r for ρ which is

Numerical estimates. Taking Case (1) as the one observationally consistent

$$A = \frac{3}{4}\frac{V}{R}$$

For stars very close to the Sun $V = 300$ Km/Sec, $R = 15,000$ parsecs (Shapley's estimate) we find for $r = 1000$ parsecs,

$$p = 15 \sin 2(\theta - \theta_0) \text{ Km/Sec} \qquad \left(A = \frac{300}{15000} \cdot \frac{3}{4} = \frac{3}{200} \text{ Km/Sec/parsec} \right)$$

and hence is considerable & easy to detect. With $\theta_0 = 325^\circ$, p is max numerically for $\theta = 10^\circ, 100^\circ, 190^\circ, 280^\circ$ and $p = 0$ for $\theta = 55^\circ, 145^\circ, 235^\circ, 325^\circ$.

Also $B = -\frac{1}{4}\frac{V}{R}$ and the proper motion is

$$\mu_\theta = \frac{V}{4R}\{3\cos 2(\theta - \theta_0) - 1\}$$

which is numerically maximum when $\theta - \theta_0 = 90^\circ$ or 270° & then

$$\mu_\theta = -\frac{V}{R} = -0''.0042, \quad \text{Also } B = -0''.0011 = -4.74 \times .0011 \text{ Km/Sec/par}$$

$\omega = V/R = 0''.0042$ per annum, leading to a period of 300 million years for stars in the neighbourhood of the Sun.

A and B can be found statistically from observations.

Dynamical results. In the two expns for K viz K_1 & K_2 although K_1 appears more consistent with observation, an expn like K_2 has to be used when the we take into account interstellar matter which undoubtedly exists (stationary H & K lines). For the latter the gravitational attractive force is to be taken in the form K_2. Thus we can write

$$K = K_1 + K_2 \qquad (1)$$

$$\text{so that} \quad A = \frac{1}{4}\frac{V}{R}\left(1 - \frac{R}{K}\frac{dk}{dR}\right) \qquad (2)$$

$$\text{with} \quad K_1 = C/R^2 \qquad (3)$$

$$K_2 = DR \qquad (4)$$

$$\text{Then (2) gives} \quad A = \frac{3}{4}\frac{V}{R}\frac{K_1}{K} \qquad (5)$$

$$B = A - \frac{V}{R} \qquad (6)$$

$$= -\frac{V}{4R}\left(\frac{K_1 + 4K_2}{K}\right) \qquad (7)$$

From (1), (5), (7), $\frac{K_1}{K} = \frac{4A}{3(A-B)}$ (8)

A & B being known from observation K_1/K can be derived

Now, $\rho = \frac{3}{4\pi} \frac{K_1}{K} \frac{V}{R} r \sin 2L$ (9)

$\rho/\rho_G = \frac{3K_1}{4K} \frac{V}{R} \left\{ \cos 2L - \frac{1}{3} - \frac{4K_2}{K_1} \right\}$ (10)

Using $\omega = V/R$ in (6), (8) gives

$\omega = \frac{4}{3} \frac{K}{K_1} A$ (11)

ie ω is determined

$K = K_1 + K_2$ ie $K > K_1$

$\therefore \omega > \frac{4}{3} A$ (12)

If $K_2 = 0$, $\omega = \frac{4}{3} A$.

Eqn (6) above gives R when we know A, B & V. With V = 300 Km/sec, R = 10,000 parsecs.

$K_1/K = 0.75$

ie central mass predominates in producing galactic rotation.

In $K_2 = D R_1$ we can write $D = \frac{G M_2}{\frac{4}{3}\pi R_1^3}$ where M_2 = mass due to interstellar matter & R_1 = boundary of galaxy.

$R_1 = 15,000$ parsecs, $\frac{R_1}{R} = \frac{3}{2}$.

An estimate of mass of galaxy can be made

we can write $A = \frac{1}{4}\omega \left(1 - \frac{1}{\omega^2} \frac{dK}{dR}\right)$ (using $K = V^2/R$

Also $K = K_1 + K_2 = \frac{G M_1}{R^2} + \frac{4}{3}\pi G \rho M_2 R$ $\boxed{\rho = 3M_2/4\pi R_1^3}$

From this $A = \frac{3GM_1}{4\omega R^3}$ ie $M_1 = \frac{4\omega R^3 A}{3G}$. (13)

further $M_2 = \dfrac{4\omega(3\omega - 4A)R_1^3}{9\pi G}$

$$\therefore \frac{M_2}{M_1} = \frac{1}{\pi}\left(\frac{R_1}{R}\right)^3\left(\frac{\omega}{A} - \frac{4}{3}\right) \quad \ldots (14)$$

$$= \frac{4}{3\pi}\left(\frac{R_1}{R}\right)^3 \frac{K_2}{K_1} \quad \ldots (15)$$

Using $A = 0.0155$ Km/Sec/parsec, $R = 10,000$ parsecs, $R_1/R = 3/2$

and $G = 4\pi^2$ ($M_\odot = 1$, $a = 1$, year $= 1$) $\therefore \omega = 300 \times 10^6$

From (13), $M_1 = 13 \times 10^{10}$

" then from (14), $M_2 = 6 \times 10^{10}$

ie $M \simeq 2 \times 10^{11}$ times the Sun's mass.

Period of rotation in the neighbourhood of the Sun, $P = 2\pi/\omega = 2 \times 10^8$ years

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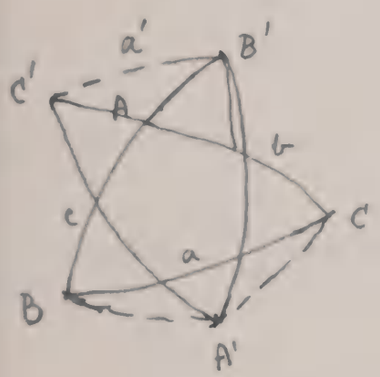
6/8/46. Spherical Trigonometry – Proof of Cagnoli's result viz.

$$\sin b \sin c + \cos b \cos c \cos A = \sin B \sin C - \cos B \cos C \cos a.$$



The following is a geometrical proof:

Produce BA & CA to B' & C' such that $BB' = CC' = 90°$. Join B'C'

Draw BA' & CA' ⊥r to BB' & CC'. Obviously B' is pole of A'B and C' of A'C so that $A'B' = A'C' = 90°$ ie A' is pole of B'C'.

From $\triangle^{le}$ AB'C', the fundamental formula gives (with $B'C' = a'$)

$$\cos a' = \sin b \sin c + \cos b \cos c \cos A \qquad (1)$$

$\angle B'A'C' = a'$, and since $B'A'B = C'A'C = 90°$, $\angle BA'C = 180 - B'A'C'$

$= 180 - a'$.

Hence the dual of the fundamental formula applied to the triangle A'BC gives

$$-\cos(180 - a') = \sin B \sin C - \cos B \cos C \cos a \qquad (2)$$

Comparing (1) & (2) proves Cagnoli's result.

Bull. Amer. Math. Soc: 54. 1948, p. 401. – Turan's inequality re. Legendre polynomials
by G. Szegö

$$\Delta_n(x) = \left\{ P_n(x)^2 - P_{n-1}(x) P_{n+1}(x) \right\} \geqq 0$$

$$n \geqq 1 \; ; \; -1 \leqq x \leqq 1 \qquad \text{(equality for } x = \pm 1\text{)}$$

[proof]:– $P_n^2 + a P_{n-1}^2 - bx P_n P_{n-1}$

$$= \left(P_n - \frac{b}{2} x P_{n-1}\right)^2 + \left(a - \frac{b^2}{4} x^2\right) P_{n-1}^2$$

This is +ve if $a > \left(\frac{bx}{2}\right)^2$ ie $\frac{n}{n+1} > \left(\frac{n+\frac{1}{2}}{n+1} x\right)^2$ $\quad x^2 < \frac{n}{n+1} \cdot \frac{(n+1)^2}{(n+\frac{1}{2})^2}$

ie $|x| < \frac{\sqrt{n(n+1)}}{n+\frac{1}{2}} = \cos\theta_0$ $\quad \left[n(n+1) < (n+\frac{1}{2})^2 \right.$
$\left.\text{ie } n < n + \frac{1}{4} \cdot \text{ie } \theta_0 \text{ is real.}\right]$

For other x ie $0 < \theta \leqq \theta_0$, using Mehler-Dirichlet formula

$$P_n(\cos\theta) = \frac{2}{\pi} \int_0^\theta \frac{\cos(n+\frac{1}{2})u}{\left\{2(\cos u - \cos\theta)\right\}^{1/2}} du \qquad [u < \theta \text{ since } \text{Lg. root is to be +ve}$$

$$\Delta_n(\cos\theta) = P_n^2(\cos\theta) - P_{n+1}(\cos\theta) P_{n-1}(\cos\theta)$$

$$= \frac{4}{\pi^2} \left[\int_0^\theta \frac{\cos(n+\frac{1}{2})u}{\left\{2(\cos u - \cos\theta)\right\}^{\frac{1}{2}}} du \cdot \int_0^\theta \frac{\cos(n+\frac{1}{2})v}{\left\{2(\cos v - \cos\theta)\right\}^{\frac{1}{2}}} dv \right.$$

$$- \frac{1}{2} \int_0^\theta \frac{\cos(n+\frac{3}{2})u}{\{\quad\}^{\frac{1}{2}}} du \int_0^\theta \frac{\cos(n-\frac{1}{2})v}{\{\quad\}^{\frac{1}{2}}} - \frac{1}{2} \int_0^\theta \cdot \frac{\cos(n-\frac{1}{2})u}{\{\quad\}^{\frac{1}{2}}} du \int_0^\theta \frac{\cos(n+\frac{3}{2})v}{\{\quad\}^{\frac{1}{2}}} dv$$

$$= \pi^{-2} \int_0^\theta \int_0^\theta (\cos u - \cos\theta)^{-\frac{1}{2}} (\cos v - \cos\theta)^{-\frac{1}{2}} \cdot \left\{ 2\cos(n+\tfrac{1}{2})u \cos(n+\tfrac{1}{2})v - \cos(n+\tfrac{1}{2})u \cos(n+\tfrac{3}{2})v \right.$$

$$\left. - \cos(n-\tfrac{1}{2})v \cos(n+\tfrac{3}{2})u \right\} du\, dv$$

Exp^n within brackets $= \cos(n+\frac{1}{2})(u+v)\left\{1 - \cos(u-v)\right\} + \cos(n+\frac{1}{2})(u-v)\left\{1 - \cos(u+v)\right\}$

First two factors in integrand is +ve and the third also will be positive [since $1 - \cos(u \pm v)$ are +ve] if $(n+\frac{1}{2})|u \pm v| \leqq \pi/2$, for then both terms in the third factor are +ve.

but $|u \pm v| \leq 2\theta$ ie $\leq 2\theta_0$

Now ~~$2\theta_0 \leq \pi/2$ for θ_0~~ a $\therefore (n+\frac{1}{2})|u \pm v| \leq (2n+1)\theta_0$

Now, $(2n+1)\theta_0 \leq \frac{\pi}{2}$ for, from $\cos\theta_0 = \frac{\sqrt{n(n+1)}}{n+\frac{1}{2}}$, we have $\sin\theta_0 = \frac{1}{2n+1}$

$\sin^2\theta_0 = 1 - \frac{n(n+1)}{(n+\frac{1}{2})^2} = \frac{1/4}{(n+\frac{1}{2})^2}$ and $\frac{\sin\theta}{\theta}$ being a decreasing fn of θ $(0 \leq \theta \leq \pi/2)$

$\frac{\sin\theta_0}{\theta_0} > \frac{2}{\pi}$ ie $\theta_0 < \frac{\pi}{2}\sin\theta_0$

ie $(2n+1)\theta_0 \leq \frac{\pi}{2}\sin\theta_0 (2n+1) \leq \pi/2$.

Second proof: Expansion of $\Delta_n(x)$ in series of Legendre polynomials

$$\Delta_n = C_0 P_0 + C_1 P_2 + \cdots + C_n P_{2n}$$ (only even terms since Δ_n contains even powers of x).

If it be proved that $C_1, C_2 \cdots C_n$ are negative, then the minimum of Δ_n is reached when $P_2, \cdots P_{2n}$ are maximum which happens for $x = 1$. But $\Delta_n(1) = 0$. Hence the result follows

If $i(a,b,c) = \int_{-1}^{+1} P_a P_b P_c \, dx$, we have

$$\frac{2}{4\nu+1} C_\nu = i(n, n, 2\nu) - i(n-1, n+1, 2\nu)$$

[Using a formula of Adams - Ferrers - F. Neumann

$i(a,b,c) = 0$, if $a+b+c$ is odd

$= 0$, " is even but no triangle with sides a, b, c exists

$= \frac{2}{2s+1} \frac{g_{s-a}\, g_{s-b}\, g_{s-c}}{g_s}$ if $a+b+c = 2s$, s an integer & a triangle with sides a, b, c exists.

Also $g_s = \frac{1 \cdot 3 \cdots (2s-1)}{2 \cdot 4 \cdots 2s}$; $g_0 = 1$]

$$= \frac{2}{2(n+\nu)+1}\left(\frac{g_\nu\, g_\nu\, g_{n-\nu}}{g_{n+\nu}} - \frac{g_{\nu-1}\, g_{\nu+1}\, g_{n-\nu}}{g_{n+\nu}}\right) \quad [\, s = n+\nu \text{ in the case}\,]$$

$(\nu \geq 1)$

Now $\frac{g_\nu}{g_{\nu-1}} = \frac{1\cdot3\cdots(2\nu-1)}{2\cdot4\cdots2\nu} \cdot \frac{2\cdot4\cdots(2\nu-2)}{1\cdot3\cdots(2\nu-3)} = \frac{2\nu-1}{2\nu}$ which is monotone

increasing
decreasing with ν (not increasing as mentioned in the article)

$$\therefore \frac{g_\nu}{g_{\nu-1}} < \frac{g_{\nu+1}}{g_\nu} \quad \text{ie} \quad g_\nu^2 < g_{\nu+1}\, g_{\nu-1}$$

Hence C_ν is negative ($\nu \geq 1$). This proves the result.

Third proof & fourth proof - are deduced from general theorems on integral fns & enable establishment of similar inequalities for Laguerre & Hermitian polynomials al

———— 0 ————

(1) $P_1^2 - P_0 P_2 = x^2 - \frac{1}{2}(3x^2-1) = \frac{1}{2}(1-x^2)$

(2) $P_2^2 - P_1 P_3 = \frac{1}{4}(3x^2-1)^2 - \frac{1}{2}x(5x^3-3x) = \frac{1}{4}\{(9x^4-6x^2+1) - 2x(5x^3-3x)\}$

$= \frac{1}{4}(1-x^2)(1+x^2)$

(3) $P_3^2 - P_2 P_4 = \frac{1}{4}(5x^3-3x)^2 - \frac{1}{16}(3x^2-1)(35x^4-30x^2+3)$

$= \frac{1}{16}\{4x^2(5x^2-3)^2 - (3x^2-1)(35x^4-36x^2+3)\}$

$= \frac{1}{16}\{4x^2(25x^4-30x^2+9) - (3x^2-1)(35x^4-30x^2+3)\}$

$= \frac{1}{16}\{-5x^6+5x^4-3x^2+3\} = \frac{1}{16}(1-x^2)(3+5x^4)$

(4) $P_4^2 - P_3 P_5 = \frac{1}{64}(35x^4-30x^2+3)^2 - \frac{1}{16}(5x^3-3x)(63x^5-70x^3+15x)$

$= \frac{1}{64}\{(1225x^8 - 2100x^6 + 1110x^4 - 180x^2 + 9)$
$- 4x^2(5x^2-3)(63x^4-70x^2+15)\}$

$= \frac{1}{64}\{1225x^8 - 2100x^6 + 1110x^4 - 180x^2 + 9$
$- 4x^2(315x^6 - 539x^4 + 285x^2 - 45)\}$

$= \frac{1}{64}(1-x^2)(9+9x^2-21x^4+35x^6)$

There only shas that $(1-x^2)$ is a factor of $\Delta_n(x)$ which is otherwise obvious. In the first three cases, it is easy to see that the remaining factor is $+^{ve}$. In (4) even this is not obvious.

$$n^2 P_{n-1}^2 + (n+1)^2 P_{n+1}^2 + 2n(n+1) P_{n-1} P_{n+1} \equiv (2n+1)^2 x^2 P_n^2$$

$$n^2 P_{n-1}^2 + (n+1)^2 P_{n+1}^2 + 2n(n+1) P_n^2 - (2n+1)^2 x^2 P_n^2 > 0.$$

Is proof by induction possible ie assuming

$$P_{n-1}^2 \geq P_{n-2} P_n$$

to prove $P_n^2 \geq P_{n-1} P_{n+1}$

$\Big\|$ $\dfrac{P_n}{P_{n-1}} > \dfrac{P_{n+1}}{P_n}$. ie $\dfrac{P_n}{P_{n-1}}$ is a decreasing fn of n

Practically nothing is known about P_n as a function of n.

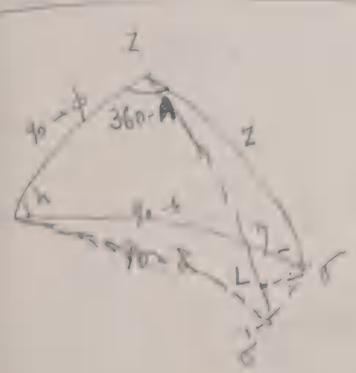
ie $\int_{n-1} \mathrm{d}P_n - P_n \, \mathrm{d}P_{n-1} < 0.$

$$P_n^2 - P_{n-1} P_{n+1} = \frac{1}{\pi^2} \left\{ \int_0^\pi \left\{ x + \sqrt{x^2-1} \cos u \right\}^n du \int_0^\pi \left\{ x + \sqrt{x^2-1} \cos v \right\}^n dv . \right.$$

$$\left. - \int_0^\pi \{ \qquad \}^{n+1} du \int \{ \qquad \}^{v} dv . \right.$$

$$\frac{1}{\pi^2} \int_0^\pi \int_0^\pi \Big[\{ \quad \}^{n-1} \{ \quad \}^n \log(\quad) - \{ \quad \}^n \{ \quad \}^{n-1} \log(\quad) \Big] d\phi$$

$$\frac{1}{\pi^2} \int\int \{ \quad \}^{2n-1} \log \frac{x + \sqrt{x^2-1} \cos\phi}{\quad} = 0 \quad (?)$$



Astronomy : To find dz/dh, d^2z/dh^2, dA/dh, d^2A/dh^2. $\left[\dfrac{\cos\delta}{-\sin A} = \dfrac{\cos\phi}{\sin\eta} \right.$

$dz = \angle\sigma' = \sigma\sigma' \sin\eta = \cos\delta \sin\eta \, dh$ ie $dz/dh = \cos\delta \sin\eta = -\cos\phi \sin A.$

$dA = \angle\sigma \operatorname{cosec} z = \sigma\sigma' \cos\eta \operatorname{cosec} z = \cos\delta \cos\eta \operatorname{cosec} z \cdot dh$

$\dfrac{dA}{dh} = \cos\delta \cos\eta \operatorname{cosec} z.$

$$\frac{d^2z}{dh^2} = -\cos\phi \cos A \cdot \frac{dA}{dh} = -\cos\phi \cos A \cos\delta \cos\eta \operatorname{cosec} z.$$

$$\frac{dA}{dh} = \cos\delta\cos\eta \Big/ \sin z. = \frac{\cos\delta}{\sin z}\left(\frac{\sin\phi - \cos z\sin\delta}{\sin z\cos\delta}\right)$$

$$= \frac{\sin\phi - \cos z\sin\delta}{\sin^2 z} = \frac{\sin\phi - \cos z(\cos z\sin\phi + \sin z\cos\phi\cos A)}{\sin^2 z}$$

$\sin\delta = \cos z\sin\phi + \sin z\cos\phi\cos A$

$$= \sin\phi - \cot z\cos\phi\cos A.$$

when due East, $\frac{dA}{dh} = \sin\phi$ ($\because A = 90$)

" Noon " $= \sin\phi$ ($\because z = \phi$)

$\frac{d^2A}{dh^2}$ Let $x = \cos z$

$$(1-x^2)\frac{dA}{dh} = \sin\phi - x\sin\delta$$

$$-2x\cdot\frac{dx}{dh}\cdot\frac{dA}{dh} + (1-x^2)\frac{d^2A}{dh^2} = -\frac{dx}{dh}\sin\delta$$

$$(1-x^2)\frac{d^2A}{dh^2} = \frac{dx}{dh}\left\{2x\frac{dA}{dh} - \sin\delta\right\}$$

$$\frac{d^2A}{dh^2} = \operatorname{cosec}^2 z\cdot\cos\phi\cos A\cdot\frac{dz}{dh} + \cot z\cos\phi\sin A\cdot\frac{dA}{dh}.$$

$$= -\operatorname{cosec}^2 z\cos\phi\cos A\cdot\cos\phi\sin A + \cot z\cos\phi\sin A(\sin\phi - \cot z\cos\phi\cos A)$$

$$=$$

$2\cos z(\sin\phi$ $2\cos z\cos\delta\cos\eta = \sin\delta.$ $2\cos z(\sin\phi - \cot z\cos\phi\cos A) = \sin\delta$

$$2\cos z(\sin\phi + \cos z\sin\delta) = \sin\delta\sin^2 z \quad 2\cos z$$

$$\cos\phi\cos A = \sin\delta\sin z - \cos\delta\cos z\cos\eta$$

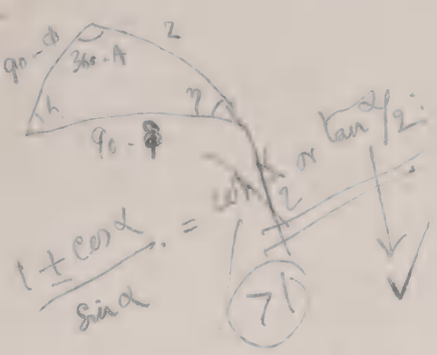
$$x = \frac{2 \pm \sqrt{4 - 4\sin^2 z}}{2\sin z}$$

$$2\cos z\sin\phi = \sin\delta\sin^2 z + 2\sin\delta\cos^2 z = \sin\delta(1+\cos^2 z).$$

$$z = \frac{1 \pm \sqrt{1 - 4\sin^2\alpha}}{2\sin\alpha}$$

$$\frac{2\cos z}{1+\cos^2 z} = \frac{\sin\delta}{\sin\phi} = \sin\alpha. \qquad x = \sin\alpha + x^2\sin\alpha$$

$x^2\sin\alpha - 2x + \sin z = 0$ // : x is known $= \tan\alpha/2$



$$\left[\frac{\sin z}{\sin\eta} = \frac{\cos\phi}{\sin A}\right] \qquad \sin\delta = \sin\phi\cos z + \cos\phi\sin z\cdot\cos A.$$

$$\cos A = \frac{\sin\delta - x\sin\phi}{\cos\phi\sqrt{1-x^2}} = \frac{(\sin\phi - x)\tan\phi}{\sqrt{1-x^2}}.$$

1)

$2\cos z(\sin\phi - \cot z\cos\phi\cos A) = \sin\delta$ $\sin z\sin\delta = 2\cos z(\sin z\sin\phi - \cos z\cos\phi\cos A)$

$z < 1$

$x = \tan^2\alpha/2$

$$\cos A = \frac{\sin\delta - x\sin\phi}{\cos\phi\sqrt{1-x^2}} = \frac{(\sin\alpha - x)\tan\phi}{\sqrt{1-x^2}}.$$

Putting $x = \tan\alpha/2$ gives the required result.

———o———

Whittaker: § 29. System rotating about an axis (z-axis)

$$T = \Sigma \tfrac{1}{2} m \{ \dot{z}^2 + \dot{r}^2 + r^2(\dot{\phi}+\omega)^2 \}$$

$$T_1 = \Sigma \tfrac{1}{2} m \{ \dot{z}^2 + \dot{r}^2 + r^2\dot{\phi} \}$$

$$W = \Sigma \tfrac{1}{2} m r^2$$

$$\frac{d\gamma(q)}{dt} = \Sigma m r^2 \dot{\phi}$$

or let $T = T_1 + \omega \frac{d\gamma}{dt} + \omega^2 W$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r$ becomes (1)

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{q}_r}\right) + \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\omega \frac{d\gamma}{dt}\right)\right\} - \frac{\partial T_1}{\partial q_r} - \frac{\partial}{\partial q_r}\left\{\omega \frac{d\gamma}{dt}\right\} - \omega^2 \frac{\partial W}{\partial q_r} = Q_r \quad \text{-- (2)}$$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\omega \frac{d\gamma}{dt}\right)\right\} = \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\omega \frac{\partial \gamma}{\partial q_s}\dot{q}_s\right)\right\} = \frac{d}{dt}\left(\omega \frac{\partial \gamma}{\partial q_r}\right)$$

$$\frac{d}{dt}\left(\frac{\partial \gamma}{\partial q_r}\right) = \frac{\partial^2 \gamma}{\partial q_r \partial q_s}\dot{q}_s$$

$$\frac{\partial}{\partial q_r}\left(\frac{d\gamma}{dt}\right) = \frac{\partial}{\partial q_r}\left(\frac{\partial \gamma}{\partial q_s}\dot{q}_s\right) = \frac{\partial^2 \gamma}{\partial q_r \partial q_s}\dot{q}_s$$

$\therefore$ (2) reduces to $\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{q}_r}\right) - \frac{\partial T_1}{\partial q_r} = -\frac{\partial}{\partial q_r}(-\omega^2 W) + Q_r$

ie. If Q_r is added on a centrifugal force whose potential is $\frac{1}{2}\Sigma m r^2 \omega^2$.

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§ 30. Quasi-Coordinates.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_K}\right) - \frac{\partial T}{\partial q_K} = Q_K \quad (\text{true co-ords. } q_K) \quad \text{.. (1)}$$

let $\omega_r = \alpha_{sr}\dot{q}_s$ .. (2) for true coordinates

& write $d\pi_r = \alpha_{sr} dq_s$ (3)

If $\frac{\partial \alpha_{sr}}{\partial q_t} = \frac{\partial \alpha_{tr}}{\partial q_s}$ (3') then π's will be true coordinates, otherwise quasi-coords

Writing (2) as $\omega_r = \alpha_{sr}\dot{q}^s$, we have solving backwards

~~$\dot{q}^k = \beta^{kt}\omega_t \quad \dot{q}^k = \beta^{kr}\omega_r$~~

Multiplying (2) $\dot{q}^k = \beta^{kt}\omega_t \quad \ldots (4)$

(1) can be written on multiplying by β^{kr} as

$$\beta^{kr}\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k}\right\} = \beta^{kr}Q_k$$

The work done in displacement $(\delta\pi_t)$ is obtained from $Q_k\delta q^k$ as $Q_k\beta^{kt}\delta\pi_t$ from (3)

$= \Pi_t\delta\pi_t$, say

$\therefore$ (1) becomes

$$\beta^{kr}\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k}\right\} = \Pi_t$$

If t be not explicitly contained in T, then using (4) we get $T(q, \dot{q}) \longrightarrow \bar{T}(\omega, q)$ and

$$\frac{\partial T}{\partial \dot{q}^k} = \frac{\partial \bar{T}}{\partial \omega_s}\cdot\frac{\partial \omega_s}{\partial \dot{q}^k} = \frac{\partial \bar{T}}{\partial \omega_s}\cdot\alpha_{ks} \text{ using (2).}$$

$$\therefore \quad \beta^{kr}\left\{\alpha_{ks}\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_s}\right) + \frac{\partial \bar{T}}{\partial \omega_s}\frac{d\alpha_{ks}}{dt} - \frac{\partial T}{\partial q^k}\right\} = \Pi_r$$

ie $\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_r}\right) + \beta^{kr}\frac{d\alpha_{ks}}{dt}\frac{\partial \bar{T}}{\partial \omega_s} - \beta^{kr}\frac{\partial T}{\partial q^k} = \Pi_r$

$$\frac{\partial T}{\partial q^k} = \frac{\partial \bar{T}}{\partial q^k} + \frac{\partial \bar{T}}{\partial \omega_s}\cdot\frac{\partial \omega_s}{\partial q^k} = \frac{\partial \bar{T}}{\partial q^k} + \frac{\partial \bar{T}}{\partial \omega_s}\frac{\partial}{\partial q^k}\left(\alpha_{ts}\dot{q}^t\right)$$

$$= \frac{\partial \bar{T}}{\partial q^k} + \frac{\partial \bar{T}}{\partial \omega_s}\frac{\partial \alpha_{ts}}{\partial q^k}\dot{q}^t$$

$\therefore$ Lagrange's eqn becomes

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_r}\right) + \beta^{kr}\frac{d\alpha_{ks}}{dt}\cdot\frac{\partial \bar{T}}{\partial \omega_s} - \beta^{kr}\frac{\partial \bar{T}}{\partial q^k} - \beta^{kr}\frac{\partial \bar{T}}{\partial \omega_s}\cdot\frac{\partial \alpha_{ts}}{\partial q^k}\dot{q}^t = \Pi_r$$

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_r}\right) + \beta^{kr}\frac{\partial \alpha_{ks}}{\partial q^m}\dot{q}^m\frac{\partial \bar{T}}{\partial \omega_s} - \beta^{kr}\frac{\partial \bar{T}}{\partial q^k} - \beta^{kr}\frac{\partial \bar{T}}{\partial \omega_s}\cdot\frac{\partial \alpha_{ms}}{\partial q^k}\dot{q}^m = 0$$

writing $\dot{q}^m = \beta^{mt}\omega_t$ and denoting $\beta^{kr}\beta^{ml}\left(\frac{\partial \alpha_{ks}}{\partial q^m} - \frac{\partial \alpha_{ms}}{\partial q^k}\right)$ by τ_{rsl}

This can be written as

$$\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \omega_r}\right) + \gamma_{rsl}\,\omega_l \frac{\partial \bar{T}}{\partial \omega_s} - \frac{\partial \bar{T}}{\partial \pi_r} = \Pi_r \qquad \ldots (5) \qquad (r = 1, 2 \ldots n$$

where $\dfrac{\partial \bar{T}}{\partial \pi_r} \equiv \rho^{kr} \dfrac{\partial \bar{T}}{\partial q^k}$.

(5) are Lagrange's eqns for quasi-coordinates. For true coordinates $\gamma_{rsl} = 0$ from (3') and we would get eqns in π_r as ordinary L-eqns

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§ 31 — generalised potential function V:

$$V = V(q, \dot{q})$$

If the eqns of motion shall be in the Lagrangian form with $L = T - V$, we should have

$$Q_r = -\frac{\partial V}{\partial q_r} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_r}\right)$$

eg: Electrodynamic case: $V = \dfrac{1}{r}\left(1 + \dfrac{\dot{r}^2}{c^2}\right)$

$$Q = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right)$$

§§ 33-34 — Similarly in dynamical systems: pattern for masses, model for forces

one system is the pattern & the other is the model

let linear dimensions be	$1 : x$
masses	$1 : y$
rate of working	$1 : z$
forces	$1 : w$

times are altered in the ratio $z : 1$

If $m\ddot{x} = X$ holds in the pattern, it must hold in the model also.

$$m(y)\left(\frac{x}{z^2}\right)\ddot{x} \neq wX$$

$$(my)(xz^{-2}\ddot{x}) = wX \qquad \therefore \quad w = xyz^{-2}$$

If $w = -1$, we have a system with forces reversed

In L-eqns introduce a time $\tau = kt$ & denote diffn w.r.t t τ by dashes

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)$ and $\frac{\partial T}{\partial q_r}$ are both homogeneous of degree $= -2$ in dt $\therefore$ L-eqns

becomes

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial q_r'}\right) - \frac{\partial \mathcal{T}}{\partial q_r} = -Q_r \qquad \text{—— (1)}$$

where $\mathcal{T}\left(q_r, q_r'\right) = T\left(q_r, \dot{q}_r\right)$

(1) is same as L-eqns with reversed forces. If $t \to it$ & initial velocities $\dot{q}_0^{(r)}$ be replaces $-i\dot{q}_0^{(r)}$ in the integrals, the roots are real & represent case of reversed forces

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§36: Lagrange's.

Routhian function R §38: Let $x, x', x'', \ldots$ be cyclic coordinates

$q_1, q_2, \ldots q_m$ palpable "

$\kappa, \kappa', \kappa'', \ldots$ cyclic momenta

$p_1, p_2 \ldots p_m$ palpable momenta.

$$R = T - \kappa\dot{x} - \kappa'\dot{x}' - \kappa''\dot{x}'' - \ldots \qquad \text{— (1)}$$

$$\delta R = \Sigma\left(p\delta\dot{q} + \frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial x}\delta x - \dot{x}\delta\kappa\right)$$

$$\left(\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial R}{\partial \dot{q}_1}\ ;\ \frac{\partial T}{\partial q_1} = \frac{\partial R}{\partial q_r}\right) \quad 2m \text{ eqns}$$

$$\text{&} \quad \left(\dot{x} = -\frac{\partial R}{\partial \kappa}\ ;\ \frac{\partial T}{\partial x} = \frac{\partial R}{\partial x}\right) \quad 2(n-m) \text{ eqns}$$

(1) can be written

$$T = R - \kappa\frac{\partial R}{\partial \kappa^2} - \kappa'\frac{\partial R}{\partial \kappa'} - \kappa''\frac{\partial R}{\partial \kappa''} - \ldots$$

$$\left[R = R\left(q, \dot{q}, \kappa\right) = A\dot{q}^2 + B\kappa^2 + C\dot{q}\kappa \text{ typical terms} \quad \text{— (2)}\right.$$

for $C\dot{q}\kappa$; $\quad R - \kappa\frac{\partial R}{\partial \kappa} = C\dot{q}\kappa - \kappa C\dot{q} = 0$ i.e. bilinear terms in $\dot{q}$ & κ vanish$\left.\right]$

i.e. $\quad T = \mathcal{T} + \mathcal{K}. \qquad \text{——— (3)}$

The m Routhian eqns can be written in the Kelvin form thus:

Let $R = R_{2,0} + R_{1,1} + R_{0,2}$ [corresp to the A, B, C types in (2)]

& $R_{1,1} = \alpha_K \dot{q}^K$ with $\alpha_K = \beta_K \eta + \beta'_K \eta' + \beta''_K \eta'' + \cdots$

(3) can be written as $T = R_{2,0} - R_{0,2}$ ie $R_{0,2} = -\mathcal{H}_0$ and $R_{2,0} = \mathcal{T}$

$\therefore R = \mathcal{T} - \mathcal{H}_0 + \alpha_K \dot{q}^K$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}^r}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_r} + \alpha_r\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_r}\right) + \frac{\partial \alpha_r}{\partial q^s}\dot{q}^s$$

$$\frac{\partial R}{\partial q^r} = \frac{\partial \mathcal{T}}{\partial q^r} - \frac{\partial \mathcal{H}_0}{\partial q^r} + \frac{\partial \alpha_K}{\partial q^r}\dot{q}^K$$

Hence the Routhian eqns becomes

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_r}\right) - \frac{\partial \mathcal{T}}{\partial q_r} + (r,s)\dot{q}^s + \frac{\partial \mathcal{H}_0}{\partial q^r} = Q_r \cdot \qquad \text{——(4)}$$

where $(r,s) = \frac{\partial \alpha_r}{\partial q^s} - \frac{\partial \alpha_s}{\partial q^r}$ $\qquad (r = 1, \cdots m)$

(4) is Kelvin's gyroscopic form of the eqns of motion.

(a) If, in particular, $\eta = \eta' = \eta'' = \cdots = 0$, $\alpha_r = 0$, $\mathcal{H}_0 = 0$. and (4) reduces to

$$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}^r}\right) - \frac{\partial \mathcal{T}}{\partial q^r} = Q_r \qquad \cdots (5)$$

(b) In (4) terms linear in $\dot{q}^s$ change sign with dt but not the other terms. Hence motion of a gyrostatic system is not reversible unless we reverse cyclic motion & also $\dot{q}_1 \cdots \dot{q}_m$ ie precessional motion of a top is not reversible unless we reverse the spin.

(c) In the case of equilibrium of a latent cyclic system, $\dot{q}_1 = \dot{q}_2 = \cdots \dot{q}_m = 0$ and (4) gives

$$Q_r = \frac{\partial \mathcal{H}_0}{\partial q^r}$$

and $\dot{x} = \frac{\partial K}{\partial \eta}$; $\dot{x}' = \frac{\partial K}{\partial \eta'}$, ..

and K can be expressed as a homogeneous quadratic in latent velocities $\dot{x}, \dot{x}', \cdots$

If this $= T_0$ then $\frac{\partial K}{\partial \dot{q}_r} = -\frac{\partial T_0}{\partial q_r}$ $\left(\bar{T} + T = b_r \dot{q}_r \text{, in general} \right.$
$\left. \text{i.e. } \frac{\partial \bar{T}}{\partial \dot{q}_r} = -\frac{\partial T}{\partial q_r} \right)$.

i.e. $Q_r = -\frac{\partial T_0}{\partial q_r}$ ..... (6)

(d) Formula (3) gives proof of ~~Bertrand's~~ & Kelvin's theorem

(i) Let the system be started from rest by given impulses of certain types & let it be constrained by means of impulses of the remaining types

Let constraints be given by $\dot{q}_1 = \dot{q}_2 = \dots = \dot{q}_m = 0$

and let $K, K', K'', \dots$ be given impulses

Bertrand

then $T = T + K_0$, $T' = K$ i.e. $T > T'$

energy of actual (unconstrained) motion is greater.

(ii) System started with prescribed $\dot{q}_1, \dots \dot{q}_m$ but otherwise free i.e. $K = K' = \dots = 0$ in the actual motion

& if constraints give rise to $K, K', \dots \neq 0$.

Energy of actual motion < constrained one

— / —

Dynamical systems of Liouville's type:

(A) $T = \frac{1}{2} v_\rho \dot{q}_\rho^2$ $\qquad v_\rho = v_\rho(q_\rho \text{ only})$.

$V = \sum \omega_\rho$

$\frac{d}{dt}(v_\rho \dot{q}_\rho) - \frac{1}{2} v_\rho' \dot{q}_\rho^2 = -\omega_\rho'$

$v_\rho \ddot{q}_\rho + \frac{1}{2} v_\rho' \dot{q}_\rho^2 = -\omega_\rho'$

$\frac{1}{2} v_\rho \dot{q}_\rho^2 + \omega_\rho = \text{Const}$ (note that $\dot{q}_\rho \frac{d\dot{q}_\rho}{dq_\rho} = \ddot{q}_\rho$

on integration w.r.t q_ρ.

(B). $T = \frac{1}{2} \rho \dot{q}_\rho^2 / v_\rho$ $= T = \frac{1}{2}(\sum u_\rho)(v_\rho \dot{q}_\rho^2)$

$V = \sum \omega_\rho / \sum u_\rho$.

Reduce to $T = \frac{1}{2} u \left(\Sigma \dot{q}_\rho^2 \right)$, $V = \Sigma \omega_\rho / u$ $\quad u = \Sigma u_\rho$

For coordinate q_ρ $\quad \frac{d}{dt}\left(u \dot{q}_\rho \right) - \frac{1}{2} \frac{\partial u}{\partial q_\rho} \left(\dot{q}_1^2 + \cdots + \dot{q}_n^2 \right) = - \frac{\partial V}{\partial q_\rho}$

Multiplying by $2 u \dot{q}_\rho$ $\quad \frac{d}{dt}\left(u^2 \dot{q}_\rho^2 \right) - u \dot{q}_\rho \frac{\partial u}{\partial q_\rho} \left(\dot{q}_1^2 + \cdots + \dot{q}_n^2 \right) = -2 u \dot{q}_\rho \frac{\partial V}{\partial q_\rho}$

but $\frac{1}{2} u \left(\dot{q}_1^2 + \cdots + \dot{q}_n^2 \right) = h - V$ from integral of energy $\quad [h = \text{const}$

$$\frac{d}{dt}\left(u^2 \dot{q}_\rho^2 \right) = 2 (h - V) \dot{q}_\rho \frac{\partial u}{\partial q_\rho} - 2 u \dot{q}_\rho \frac{\partial V}{\partial q_\rho}$$

$$= 2 \dot{q}_\rho \frac{\partial}{\partial q_\rho} \left\{ (h - V) u \right\} = 2 \dot{q}_\rho \frac{\partial}{\partial q_\rho} \left\{ h u_\rho - \omega_\rho \right\} \quad \text{(no summation for } \rho\text{)}$$

$$= 2 \frac{d}{dt} \left\{ h u_\rho - \omega_\rho \right\}$$

Integrating $\frac{1}{2} u^2 \dot{q}_\rho^2 = h u_\rho - \omega_\rho + \gamma_\rho$, where $\Sigma \gamma_\rho = 0$

a. $\left(h u_\rho - \omega_\rho + \gamma_\rho \right)^{-\frac{1}{2}} dq_\rho = \text{const}$

$\left(h u_1 - \omega_1 + \gamma \right)^{-\frac{1}{2}} dq_1 = \text{etc.}$ on integration give the solution

---

§ 77. $\quad T = \frac{1}{2} a_{rs} \dot{q}_r \dot{q}_s$; $V = \frac{1}{2} b_{rs} q_r q_s$

Let $Q = h_r q^r$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r} \right) = - \frac{\partial V}{\partial q_r} \quad \text{i.e.} \quad a_{rs} \ddot{q}^s = - b_{rs} q^s$$

multiplying by m_r, $m_r a_{rs} \ddot{q}^s = - m_r b_{rs} q^s$.

$$\frac{d}{dt}\left(a_{rs} \dot{q}^s \right) = - b_{rs} q^s \quad \text{i.e.} \quad a_{rs} \ddot{q}_s = - b_{rs} q_s$$

$$m_r a_{rs} \ddot{q}_s = - m_r b_{rs} q_s$$

If $m_r b_{rs} = \lambda . m_r a_{rs} = \lambda h_s$, and $Q = h_s q_s$

this gives $\lambda h_s \ddot{q}^s = - \lambda^2 h_s q^s$ i.e. $\ddot{Q} + \lambda Q = 0$

It is possible to find a set m_s & h_s if $\|\lambda a_{rs} - b_{rs}\| = 0$

Results assumed (1) By successive reduction the two quadratic forms can be brought to the form $T = \frac{1}{2}(\dot{q}_1^2 + \cdots + \dot{q}_n^2)$

$$V = \frac{1}{2}(\mu_1 q_1^2 + \cdots + \mu_n q_n^2).$$

These q_1's are called normal coordinates.

$\mu_1, \mu_2 \cdots \mu_n$ are roots of $\|\lambda a_{rs} - b_{rs}\| = 0$ in view of its invariantive character.

(2) Roots of determinantal eqn are real when T is a +ve definite form ie transformation to normal coords is always real.

(3) If V also be +ve definite, the roots are all +ve.

(4) normal modes & superposition of normal modes.

9/4/52 Effect of ~~ex~~ constraint:-

Vibration about steady motion: Consider a system containing ignorable coordinates. The modified Routhian equations reduce to Kelvins gyroscopic eqns which can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \sum_{s=1}^{m}(r,s)\dot{q}_s + \frac{\partial \mathcal{K}_0}{\partial q_r} = P_r \qquad \text{——(1)}$$

($r = 1, \cdots m$ gives the non-ignorable coordinates)

To investigate free motions we put $P_r = 0$. Also writing

$$T = \frac{1}{2} a_{rs} \dot{q}_r \dot{q}_s$$

with a_{rs} as constants in case of small oscillations, we have $\frac{\partial T}{\partial \dot{q}_r} = 0$.

Also we can write $\mathcal{K}_0 = \frac{1}{2} c_{rs} q_r q_s$, where c_{rs} depends on the values of the cyclic momenta

Eqm position is given by $\frac{\partial \mathcal{K}_0}{\partial q_r} = 0$ & we will assume that $\mathcal{K}_0$ is actually a minimum in the eqm configuration.

We can bring T & $\mathcal{K}$ simultaneously to the forms

$2T = \Sigma a_r \dot{q}_r^2$, $2\mathcal{K}_0 = \Sigma c_r q_r^2$. Replace (r,s) by β_{rs} - (1) becomes

$$\left.\begin{aligned} a_1\ddot{q}_1 + c_1 q_1 \quad &+ \beta_{12}\dot{q}_2 + \beta_{13}\dot{q}_3 + \cdots + \beta_{1n}\dot{q}_n = 0 \\ a_2\ddot{q}_2 + c_2 q_2 + \beta_{21}\dot{q}_1 \quad &+ \beta_{23}\dot{q}_3 + \cdots + \beta_{2n}\dot{q}_n = 0. \\ \cdots\cdots\cdots\

& roots occur in pairs $\lambda = \pm i\sigma$

Minors $\alpha_1, \alpha_2 \ldots \alpha_n$ will involve odd as well as even powers of λ & can't therefore be Compl

Putting $\lambda = i\sigma$, $\alpha_r = \mu_r + i\nu_r$

$$q_r = C(\mu_r + i\nu_r)e^{i\sigma t} = He^{i\epsilon}(\mu_r + i\nu_r)e^{i\sigma t} \quad [\text{putting } C = He^{i\epsilon}$$

Taking the real part, $q_r = H\{\mu_r \cos(\sigma t + \epsilon) - \nu_r \sin(\sigma t + \epsilon)\}$, H & ϵ, arbitrary

This represents a "natural mode" of oscillation.

Examples on free rotation & top motion:

(1) Integrate Euler's eqns in the cases $A = B$ & $H^2 = 2BT$.

(a) $A = B$: Here $\dot{\omega}_3 = 0$ i.e. $\omega_3 = \text{Const} = \underline{n}$. Writing $A - C = A\lambda$, the first two of

Euler's eqns become $\dot{\omega}_1 - \lambda n \omega_2 = 0$, and $\dot{\omega}_2 + \lambda n \omega_1 = 0$

i.e. $\ddot{\omega}_1 + \lambda^2 n^2 \omega_1 = 0$, giving $\omega_1 = E\cos(\lambda n t + \alpha)$
& $\omega_2 = -E\sin(\lambda n t + \alpha)$

$$A\omega_1 = -H\sin\theta\cos\psi, \quad A\omega_2 = H\sin\theta\sin\psi$$

$$\therefore \tan\psi = -\frac{\omega_2}{\omega_1} = \tan(\lambda n t + \alpha) \quad \text{i.e. } \psi = \lambda n t + \alpha$$

Also from Euler's geometrical eqns $\dot{\theta} = 0$, and $\dot{\phi} = H/A$.

(b) $H^2 = 2BT$: $A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T$

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2 = 2BT.$$

$$\therefore \frac{\omega_1^2}{C(C-B)} = \frac{2T - B\omega_2^2}{AC(C-A)} = \frac{\omega_3^2}{A(B-A)}$$

so that Euler's second eqn viz. $B\dot{\omega}_2 = (C-A)\omega_3\omega_1$ gives

$$B\dot{\omega}_2 = \sqrt{\left\{\frac{(B-A)(C-B)}{AC}\right\}} \cdot (2T - B\omega_2^2) \quad \text{or} \quad \dot{\omega}_2 = \mu(q^2 - \omega_2^2), \text{ say } \left(q^2 = \frac{2T}{B}\right)$$

Giving $\frac{\omega_2}{q} = \frac{Fe^{2\mu q t} - 1}{Fe^{2\mu q t} + 1}$

As $t \to \infty$, $\omega_2 \to q\ (= \sqrt{2T/B})$ & then $\omega_2 = q$, $\omega_1 = \omega_3 = 0$ i.e. ultimately

the axis of rotation coincides with the mean axis

(2) If principal moments of inertia be $(6A, 3A, A)$ & initially $\omega_1 = n$, $\omega_2 = 0$, $\omega_3 = 3n$, Show that at any later time

$$\omega_2 = -\sqrt{5}\, n \tanh(\sqrt{5}\, nt).$$

& show that ultimately the above result of Ex (1) holds.

$$6\omega_1^2 + 3\omega_2^2 + \omega_3^2 = 15n^2 \quad \text{(constancy of energy)}$$

$$36\omega_1^2 + 9\omega_2^2 + \omega_3^2 = 45n^2 \quad (\ \ \text{"} \quad \text{A.M})$$

$$\therefore \quad \omega_1^2 = \frac{\omega_2^2 - 5n^2}{-5} = \frac{\omega_3^2}{9} \qquad \text{——— (1)}$$

By Euler's second eqn $3\dot{\omega}_2 + 5\omega_3\omega_1 = 0$ or $\dot{\omega}_2 = -(5n^2 - \omega_2^2)$

& using the initial cons^n $\omega_2 = 0$, this gives req^d result on integration.

As $t \to \infty$, $\omega_2^2 \to 5n^2$ so that from (1), $\omega_1, \omega_3 \to 0$. Hence result stated follows

3) Obtain the condition for steady motion of a top.

Denoting by Ω the angular velocity about OZ in steady motion with $\theta = \alpha$, we have the equation

$$0 = -b\dot{\phi} + A\dot{\phi}^2 \cos\theta + Mgh$$

written as $Cn\Omega - A\Omega^2 \cos\alpha - Mgh = 0$

giving two possible rates of precession. The cond^n for real roots of Ω is

$$C^2n^2 \geq 4AMgh\cos\alpha.$$

(4) Investigation the motion of a top starting with a given spin $\underline{n}$ and its axis at an angle α to the vertical.

Here initial conditions are $\dot{\theta} = 0$, $\dot{\phi} = 0$, $\theta = \alpha$. & the eq^{ns}

$$A\dot{\phi}\sin^2\theta + C(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = a$$

gives $A\dot{\phi}\sin^2\theta + C(n + \dot{\phi}\cos\theta)\cos\theta = Cn\cos\alpha$

$$C(\dot{\psi} + \dot{\phi}\cos\theta) = b = Cn \text{ i.e. } \dot{\psi} + \dot{\phi}\cos\theta = n$$

Now $A\dot{\phi}\sin^2\theta + Cn\cos\theta = Cn\cos\alpha$

ie $A\dot\phi \sin^2\theta = Cn(\cos\alpha - \cos\theta)$ — — (1)

Also the eqn $\frac{1}{2}A\dot\theta^2 = -\frac{(a-b\cos\theta)^2}{2A\sin^2\theta} - Mgh\cos\theta + c$

Can be written, by writing $a - b\cos\theta = A\dot\phi\sin^2\theta$, as

$$A(\dot\phi^2\sin^2\theta + \dot\theta^2) + 2Mgh\cos\theta = C$$

& finis $A(\dot\phi^2\sin^2\theta + \dot\theta^2) + 2Mgh\cos\theta = 2Mgh\cos\alpha$ ——(2)

Eliminating $\dot\phi$ from (1) & (2), and putting $C^2n^2 = 4\lambda MghA$, we see from the eqn for

that results that $\dot\theta = 0$ for $\theta = \alpha$ and also when

$$\cos^2\theta - 2\lambda\cos\theta + 2\lambda\cos\alpha - 1 = 0$$

$$\text{or } \cos\theta = \lambda \pm \sqrt{(1 - 2\lambda\cos\alpha + \lambda^2)} \quad — (3)$$

Taking the + sign we get $\cos\theta > 1$. Hence we take the − sign

$$\text{ie } \cos\theta = \lambda - \sqrt{(1 - 2\lambda\cos\alpha + \lambda^2)} = \cos\beta, \text{ say}$$

ie axis of top oscillates between $\theta = \alpha$ & $\theta = \beta$.

(5) Acrobat.

Gyrostat. A solid of revolution rotating about a rt: rod coincident with its axis is a gyrostat. Its C.G. lies on the axis and $A = B$. ~~External ab~~ The external forces have no moment about the axis & hence spin is const

Ex (5) A top is precessing steadily with one end of its axis fixed, and axis horizontal. If a small weight w be suddenly attached to the other end of its axis, without impulse, show that an oscillatory motion is set up in which the maximum dip of the axis below the horizontal is approximately $2wAa/(A^2\Omega^2 + C^2n^2)$, where a = length of axis.

For steady motion with $\alpha = \pi/2$

$$Cn\Omega = Mgh \quad — — (1)$$

From eqn of Constancy of A.M

$$A\dot{\phi}\sin^2\theta + Cn\cos\theta = A\Omega \quad (\alpha = 90^\circ) \quad \text{--- (2)}$$

the eqn for θ becomes in the disturbed motion

$$A\ddot{\theta} + Cn\dot{\phi}\sin\theta - A\dot{\phi}^2\sin\theta\cos\theta = (Mgh + wa)\sin\theta \quad \text{--- (3)}$$

Eliminating $\dot{\phi}$ & putting $\theta = \pi/2 + \chi$, this gives (retaining terms up to first power of χ)

$$A^2\ddot{\chi} + (A^2\Omega^2 + C^2n^2)\chi = waA$$

Putting $\chi = \dfrac{waA}{A^2\Omega^2 + C^2n^2} = \xi$, we get req$^{\text{d}}$ result.

(6) Investigate small oscillations about steady motion in the pendulum general case ($\alpha \neq 90^\circ$)

The eqn for θ is

$$A\ddot{\theta} + Cn\dot{\phi}\sin\theta - A\dot{\phi}^2\sin\theta\cos\theta = Mgh\sin\theta \quad \text{——— (1)}$$

Eqn of A.M is $\quad A\dot{\phi}\sin^2\theta + Cn\cos\theta = A\Omega\sin^2\alpha + Cn\cos\alpha \quad \text{——— (2)}$

For steady motion $\quad Cn\Omega - A\Omega^2\cos\alpha = Mgh \quad \text{——— (3)}$

Eliminating $\dot{\phi}$ from (1) & (2) gives

$$A\ddot{\theta} + \frac{Cn(b - Cn\cos\theta)}{A\sin\theta} - \frac{(b - Cn\cos\theta)^2\cos\theta}{A\sin^3\theta} = Mgh\cos\theta \qquad (b = A\Omega\sin^2\alpha + Cn\cos\alpha)$$

Putting $\theta = \alpha + \chi$ & proceed as in Eq(5) above, we get, after some simplification

$$A\ddot{\chi} + \chi\left(M^2g^2h^2 - 2MghA\Omega^2\cos\alpha + A^2\Omega^4\right)/A\Omega^2 = 0$$

Hence period of oscillation is given.

7) <u>The sleeping top</u>: top with steady motion with $\alpha = 0$. To find oscillations about steady motion

$$T = \tfrac{1}{2}A\dot{\theta}^2 + \tfrac{1}{2}A\dot{\phi}^2\sin^2\theta + \tfrac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

$$V = Mgh\cos\theta$$

$C(\dot{\psi} + \dot{\phi}\cos\theta) = b$ is one integral

and hence after ignoration of ψ, we get

$$R = \tfrac{1}{2} A\dot\theta^2 + \tfrac{1}{2} A\dot\phi^2 \sin^2\theta + b\dot\phi\cos\theta - Mgh\cos\theta$$

or $$R = \tfrac{1}{2} A\dot\theta^2 + \tfrac{1}{2} A\dot\phi^2 \sin^2\theta + b\dot\phi(\cos\theta - 1) - Mgh(\cos\theta - 1)$$ [$b\dot\phi$ + Mgh are Constants]

[See Whittaker's Analytical Dynamics. p. 206]

Other examples.

(1) A uniform rod is in motion under the action of gravity alone. Show that the equation for determining its inclination θ to the vertical at time t is

$$\left(\frac{d\theta}{dt}\right)^2 = a - b\operatorname{cosec}^2\theta,$$ where a and b are constants.

(2) A ring of wire of radius c rests on the top of a smooth fixed sphere of radius a, and is set rotating about the vertical diameter of the sphere. If the ring is slightly displaced prove that the motion is unstable if the angular velocity is less than

$$\left\{ 2g \frac{2a^2 - c^2}{c^4} \sqrt{(a^2 - c^2)} \right\}^{1/2}.$$

(3) A top moves under gravity with the peg fixed. Show that the equations to determine the motion of the axis are of the form

$$\dot x^2 = (1-x^2)(e - 2qx) - (h - px)^2 \;;\quad (1-x^2)\dot\psi = h - px$$

where $x = \cos\theta$, and p, q, e, h are constants. Show that if $p^2 = 3q$ and if initially $\theta = \pi/3$, $\dot\theta = 0$, $\dot\psi = \frac{2}{3}p$, then $e = 2q$, $h = p$, and integrate the equations in this case.

(4) A gyrostat has a point in its axis attached by a universal joint and a string of length l to a fixed point; an angular velocity ω is given to the body, and n is the angular velocity of the centre of gravity about the vertical, θ the angle which the string, ϕ that which the axis of the body makes with the vertical, a the distance between the centre of gravity and the point of suspension; C being the moment of inertia about the axis of the figure. Show that when in steady motion

$$n\sin\phi\,(C\omega - An\cos\phi)\cos\theta = Mga\sin(\phi - \theta) \;;\quad n^2(l\sin\theta + a\sin\phi) = g\tan\theta.$$

(5) If a constant couple be applied about the axis of symmetry of a body supported at its centre of inertia and initially rotating about an axis perpendicular to that of symmetry, determine the motion completely; and show that the cone described in the body by the instantaneous axis has the equation

$$\tan^{-1}\left(\frac{x}{y}\right) = \frac{A-C}{A} \cdot \frac{\Omega^2 C}{2N} \cdot \frac{z^2}{x^2+y^2}$$

where N is the couple, Ω the initial angular velocity.

(6) The centre O of a sphere is fixed at a point P of the earth's surface and a radius OC is constrained to move in the meridian plane at P. Initially OC is parallel to the axis of the earth and the sphere has an angular velocity $\underline{n}$ about OC. Prove that if the sphere be disturbed, OC will oscillate in the meridian in a time

$$2\pi \Big/ \sqrt{(n\omega - \omega^2)},$$

where $\underline{\omega}$ is the angular velocity of the Earth about its axis.

(7) A rigid body rotates freely about its centre of mass under no external forces. The body is rotating with angular velocity $\underline{\omega}$ about the axis of greatest moment (C) when a small angular momentum $\underline{Ap}$ is applied about the axis of least moment (A). Show that the instantaneous axis moves in the body with period $2\pi/\gamma$, where

$$AB\gamma^2 = (C-B)(C-A)\omega^2.$$

(8) A solid cube is in motion about an angular point which is fixed. If there are no external forces, and $\omega_1, \omega_2, \omega_3$ are the angular velocities about the edges through the fixed point, prove that $\omega_1 + \omega_2 + \omega_3$ and $\omega_1^2 + \omega_2^2 + \omega_3^2$ are each constant.

(9) For a body moving about a fixed point under no forces prove

(i) $\omega \dfrac{d\omega}{dt} = \sqrt{\{(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)(\omega_3^2-\omega^2)\}}$, where $\omega_1^2 = \dfrac{2T(B+C)-H^2}{BC}$, etc

& deduce that if $A > B > C$, the maximum value of ω is ω_2.

(ii) $A^3p^2 + B^3q^2 + C^3r^2 = ABC\omega^2 - 2(BC+CA+AB)T + (A+B+C)H^2$.

(10) A top is spinning very rapidly about its axis, which is inclined at an angle α to the vertical, and momentarily at rest. Prove that the axis will descend through an angle

$$\frac{2A\,Mgh}{C^2n^2}\sec\alpha, \text{ approximately.}$$

———— 0 ————

Prof Pauli's Theory of Relativity.

p.30 – Proof of (46) – might be done by direct multiplication and cancellation of terms. Perhaps a more elegant method would be to consider the determinant

$$\Delta = \begin{vmatrix} x_1 & y_1 & x_1 & y_1 \\ x_2 & y_2 & x_2 & y_2 \\ x_3 & y_3 & x_3 & y_3 \\ x_4 & y_4 & x_4 & y_4 \end{vmatrix} = 0$$

and expand it by Laplace's method getting in terms of first two columns

$$\Delta = 2\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \cdot \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_1 & x_3 \\ x_2 & y_1 \end{vmatrix} \cdot \begin{vmatrix} x_2 & y_2 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix} \cdot \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \right] = 0$$

ie $\zeta^{12}\zeta^{34} + \zeta^{31}\zeta^{24} + \zeta^{14}\zeta^{23} = 0$.

p.33 – To show that in view of (53, a) and 53(b), the independent components of b_{iklm} reduces to $\frac{1}{12}n^2(n^2-1)$.

K. L. Malhotra
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2) H. Bondi – Cosmology, 1960

3) E. Hubble – Observational approach to cosmology, 1937

4) W. Bonnor – The mystery of the expanding Universe, 1964

5) J. D. North – The measure of the Universe, Oxford, 1965 (A history of modern Cosmology). } This book which contains a huge bibliography.

} All in cosmogony Section

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